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ANALYSIS

-problems, solutions and hints-

Math 1: **Analysis**
Problems, Solutions and Hints

FOR THE ELECTRONICS AND TELECOMMUNICATION STUDENTS

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Preface

This is the complementary text to my Calculus Lecture Notes for the Electronics and Telecommunication students at Technical University in Poznań. It is an outgrowth of my teaching of Calculus I at Technical University of Poznań (for the first year students).

The goal of this text is to help students learn to use the most difficult parts of calculus intelligently in order to be able to solve a wide variety of mathematical and physical problems. The exercise sets have been carefully constructed to be of maximum use to the students.

Prerequisite material from algebra, trigonometry, and analytic geometry is consistent with the Polish standards. Students are advised to assess themselves and to take a pre-calculus course if they lack the necessary background.

The author de-emphasize the theory of limits, leaving a detailed study to the end of the course, after the students have mastered the fundamentals of calculus-differentiation and integration.

Computer and calculator applications are used for motivation and to illustrate the numerical content of calculus. In my view the ability to visualize basic graphs and to interpret them mentally is very important in calculus and in subsequent mathematics courses.

This text leaves out the less important parts of the course because of the limited capacity of the book.

Misprints are a plague to authors (and readers) of mathematical textbooks. The author have made a special effort to weed them out, and we will be grateful to the readers who help us eliminate any that remain.

Andrzej Maćkiewicz
Poznań, September 2014

1

Logic and techniques of proof (Exercises)

1.1 Practice Problems

Example 1 *Prove that $x^n - y^n$ has $x - y$ as a factor for all positive integers n.*

Solution: The statement is true for $n = 1$ since $x^1 - y^1 = x - y$. Assume the statement true for $n = k$, i.e., assume that $x^k - y^k$ has $x - y$ as a factor. Consider

$$\begin{aligned} x^{k+1} - y^{k+1} &= x^{k+1} - x^k y + x^k y - y^{k+1} \\ &= x^k(x - y) + y(x^k - y^k). \end{aligned}$$

The first term on the right has $x - y$ as a factor, and the second term on the right also has $x - y$ as a factor because of the above assumption. Thus $x^{k+1} - y^{k+1}$ has $x - y$ as a factor if $x^k - y^k$ does. □

Example 2 *If $n \in \mathbb{Z}$ and $n \geq 0$, then $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.*

Solution: We will prove this with mathematical induction.

1) If $n = 0$, this statement is

$$\sum_{i=0}^{0} i \cdot i! = (0+1)! - 1$$

Since the left-hand side is $0 \cdot 0! = 0$, and the right-hand side is $1! - 1 = 0$ the equation $\sum_{i=0}^{0} i \cdot i! = (0+1)! - 1$ holds, as both sides are zero.

2) Consider any integer $k \geq 0$. We must show that S_k implies S_{k+1}. That is, we must show that

$$\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$$

implies

$$\sum_{i=0}^{k+1} i \cdot i! = ((k+1)+1)! - 1.$$

We use direct proof. Suppose $\sum_{i=0}^{k} i{\cdot}i! = (k+1)! - 1$.. Observe that

$$\begin{aligned}\sum_{i=0}^{k+1} i{\cdot}i! &= \sum_{i=0}^{k} i{\cdot}i! + (k+1)\,(k+1)! \\ &= ((k+1)! - 1) + (k+1)\,(k+1)! \\ &= (1 + (k+1))\,(k+1)! - 1 \\ &= (k+2)! - 1 \\ &= ((k+1)+1)! - 1,\end{aligned}$$

Therefore

$$\sum_{i=0}^{k+1} i{\cdot}i! = ((k+1)+1)! - 1.$$

It follows by induction that

$$\sum_{i=0}^{n} i{\cdot}i! = (n+1)! - 1$$

for every integer $n \geq 0$.

Example 3 *If $n \in \mathbb{N}$, then $(1+x)^n \geq 1 + nx$ for all all $x \in \mathbb{R}$ with $x > -1$.*

Solution: We will prove this with mathematical induction.

1) For the basis step, notice that when $n = 1$ the statement is $(1+x)^1 \geq 1+1\,x$, and this is true because both sides equal $1 + x$.

2) Assume that for some $k \geq 1$, the statement $(1+x)^k \geq 1 + kx$ is true for all all $x \in \mathbb{R}$ with $x > -1$. From this we need to prove

$$(1+x)^{k+1} \geq 1 + (k+1)x.$$

Now, $1 + x$ is positive because $x > -1$, so we can multiply both sides of $(1+x)^k \geq 1 + kx$ by $(1+x)$ without changing the direction of the inequality $\geq$.

$$\begin{aligned}(1+x)^k(1+x) &\geq (1+kx)(1+x), \\ (1+x)^{k+1} &\geq 1 + x + kx + kx^2, \\ (1+x)^{k+1} &\geq 1 + (k+1)x + kx^2\end{aligned}$$

The above term kx^2 is positive, so removing it from the right-hand side will only make that side smaller. Thus we get $(1+x)^{k+1} \geq 1 + (k+1)x$.

1.2 Exercises

Exercise 1.1 *The* Fibonacci numbers $\{F_n\}_{n=1}^{\infty}$ *are defined by* $F_1 = F_2 = 1$, *and*

$$F_{n+1} = F_n - F_{n-1}, \quad n \geq 2.$$

Prove by induction that

$$F_n = \frac{\left(1+\sqrt{5}\right)^n + \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}, \quad n \geq 1.$$

Exercise 1.2 *Thus, the first several Fibonacci numbers are* $F_1 = 1$; $F_2 = 1$; $F_3 = 2$; $F_4 = 3$; $F_5 = 5$; $F_6 = 8$; $F_7 = 13$; $F_8 = 21$; et cetera. *Use mathematical induction to prove the following formula involving Fibonacci numbers.*

$$\sum_{i=1}^{n} (F_i)^2 = F_n \cdot F_{n+1}.$$

Exercise 1.3 *Let the numbers* $a_0, a_1, a_2, \ldots$ *be defined by*

$$a_0 = 1, \;\; a_1 = 3, \ldots, a_n = 4(a_{n-1} - a_{n-2}) \quad \textit{for} \quad n \geq 2.$$

Show by induction that $a_n = 2^{n-1}(n+2)$ *for all* $n \geq 0$.

Exercise 1.4 *Prove by induction that*

$$\begin{aligned}(x_1 + x_2 + \ldots + x_k)^2 \quad &= \quad \sum_{i=1}^{k} x_i^2 + 2(x_1x_2 + x_1x_3 + \ldots + x_1x_k + x_2x_3 \\ &\quad\quad + x_2x_4 + \ldots x_2x_k + x_3x_4 + x_3x_5 + \ldots + x_{k-1}x_k).\end{aligned}$$

Exercise 1.5 *Prove by induction that*

$$\sin x + \sin 3x + \sin 5x + \ldots + \sin(2n-1)x = \frac{1 - \cos 2nx}{2\sin x}, \quad \textit{for} \quad n \geq 1.$$

HINT: You will need trigonometric identities that you can derive from the identities

$$\begin{aligned}\cos(\alpha - \beta) \quad &= \quad \cos\alpha\cos\beta + \sin\alpha\sin\beta, \\ \cos(\alpha + \beta) \quad &= \quad \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned}$$

Take these two identities as given.

Exercise 1.6 *Show that*

$$\sqrt{2+\sqrt{2+\sqrt{2+...\sqrt{2}}}} = 2\cos\frac{\pi}{2^{n+1}}$$

where there are n 2s in the expression on the left. `HINT`*: We know that for any angle θ we have:*

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}.$$

Exercise 1.7 *Prove by induction that, for all $x \neq 1$,*

$$1+2x+3x^2+...+nx^{n-1} = \frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}.$$

Exercise 1.8 *Prove by induction that for any positive integer n*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+...+\frac{1}{2^{n-1}} \geq \frac{1}{2}(n+1).$$

`HINT`*: Compare with Example (**??**).*

Exercise 1.9 *A* "postage stamp problem" *is a problem that (typically) asks us to determine what total postage values can be produced using two sorts of stamps. Suppose that you have 3¢ stamps and 7¢ stamps, show (using strong induction) that any postage value 12¢ or higher can be achieved. That is,*

$$\text{for any } n \in \mathbb{N}, n \geq 12 \Rightarrow \exists\ x, y \in \mathbb{N}, \quad n = 3x+7y.$$

Show that any integer postage of 12¢ or more can be made using only 4¢ and 5¢ stamps.

Exercise 1.10 *Suppose that m and n are integers, with $0 \leq m \leq n$. The* binomial coefficient $\binom{n}{m}$ *is the coefficient of t^m in the expansion of $(1+t)^n$; that is,*

$$(1+t)^n = \sum_{m=0}^{n} \binom{n}{m} t^m.$$

From this definition it follows immediately that

$$\binom{n}{0} = \binom{n}{n} = 1, \quad n \geq 0.$$

For convenience we define

$$\binom{n}{-1} = \binom{n}{n+1} = 0, \quad n \geq 0.$$

a) *Show that*

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}, \quad 0 \le m \le n,$$

and use this to show by induction on n that

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}, \quad 0 \le m \le n.$$

b) *Show that*

$$\sum_{m=0}^{n} (-1)^m \binom{n}{m} = 0 \quad \textit{and} \quad \sum_{m=0}^{n} \binom{n}{m} = 2^n.$$

c) *Show that*

$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m}.$$

(This is the binomial theorem.*)*

`HINT:` *This is certainly true for $n = 1$, since*

$$(x+y)^1 = x + y = \binom{1}{0} x + \binom{1}{1} y.$$

When $n = k + 1$, write

$$\begin{aligned} (x+y)^{k+1} &= (x+y)\,(x+y)^k \\ &= (x+y) \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j} \\ &= \sum_{j=0}^{k} \binom{k}{j} x^j y^{k+1-j} + \sum_{j=0}^{k} \binom{k}{j} x^{j+1} y^{k-j} \end{aligned}$$

and replace j by $j-1$ in the last sum to obtain

$$(x+y)^{k+1} = y^{k+1} + \sum_{j=1}^{k} \left[\binom{k}{j} + \binom{k}{j-1} \right] x^j y^{k+1-j} + x^{k+1}.$$

Finally, show how the right-hand side here becomes

$$y^{k+1} + \sum_{j=1}^{k} \binom{k+1}{j} x^j y^{k+1-j} + x^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} x^j y^{k+1-j}.$$

The identity

$$\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j} \qquad (k \geq j \geq 1)$$

is usually called the Pascal Triangle ldentity, *since in the triangle of numbers*

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & \mathbf{4} & & \mathbf{6} & & 4 & & 1 & \\
1 & & 5 & & \mathbf{10} & & 10 & & 5 & & 1
\end{array}$$

each "interior" entry is the sum of the two entries above it, and since the n-th row turns out to be $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. Thus, if for example $n = 5$, then directly from the Pascal's triangle we have

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

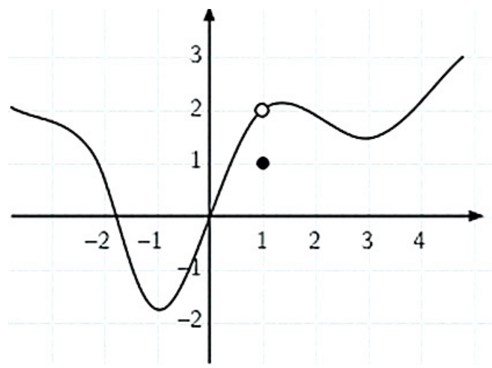
Exercise 1.11 *Show that $\sqrt{3}$ and $\sqrt[3]{2}$ are not rational.*

2

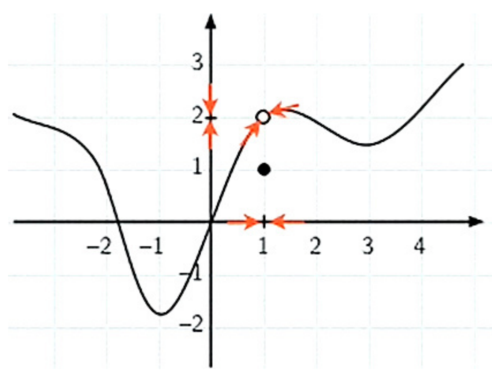
Introduction to limits (Exercises)

2.1 Basic concepts

Suppose that a function f has the graph shown below.



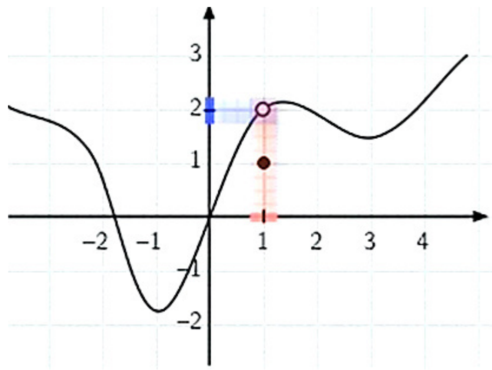
Our goal is to describe the behavior of f in the vicinity of $x = 1$ in a concise manner. Notice that $f(1) = 1$. Yet, if $x \approx 1$, then $f(x) \approx 2$.



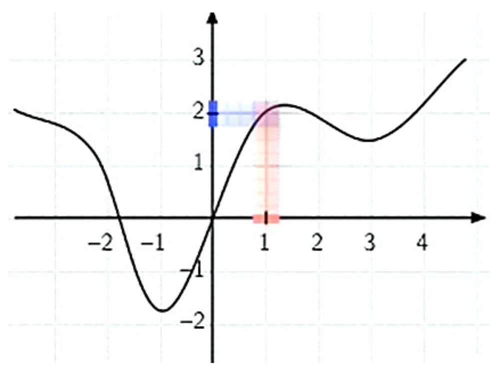
So, the number 2 is crucial in describing the behavior of f *near* 1. We say that 2 is the *limit* of $f(x)$ as x approaches 1. This is written compactly as

$$\lim_{x \to 1} f(x) = 2.$$

To be more precise, the reason that the limit is 2 as x approaches 1 is that for any interval centered at 2 in the y-axis (no matter how small) the number $f(x)$ will be in that interval for all x other than 1 in some significantly small interval centered at $x = 1$ in the x-axis.

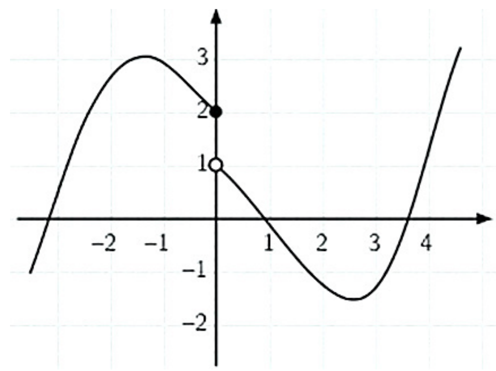


Also we point out, that $\lim_{x\to 1} f(x)$ has nothing to do with the value of f at 1.We can change $f(1)$ to any number we want, or even leave it undefined and the limit remains 2. Notice, that if $\lim_{x\to 1} f(x) = 2$ is different than $f(1)$ there is a "hole" in the graph at $(1, 2)$.



If $f(1)$ were equal to 2, the "hole" would be filled. Value and limit coincide, whenever the graph of f is continuous. This idea the basis of the mathematical definition of continuity (that will be presented later).

Let us look at another example. Again suppose that $f(x)$ has the graph shown below. Here the interesting behavior of f is in the vicinity of $x = 0$. Notice that $f(0) = 2$. If $x \approx 0$ and $x < 0$ then $f(x) \approx 2$. But if $x \approx 0$ and $x > 0$ then $f(x) \approx 1$. Therefore, the limit of $f(x)$ as x approaches 0 *does not exist.*



However, we can say that 2 is the *limit* of $f(x)$ as x approaches 0 *from the left* and express this by writing

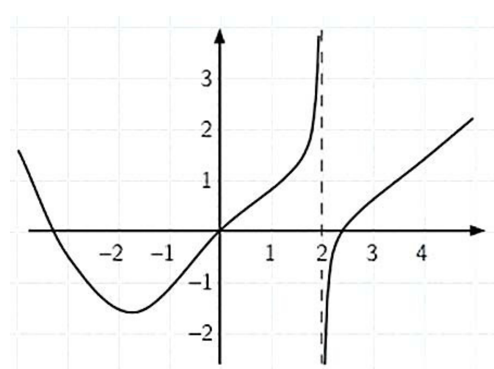
$$\lim_{x \to 0^-} f(x) = 2.$$

We can also say, that 1 is is the *limit* of $f(x)$ as x approaches 0 *from the right* and express this by writing

$$\lim_{x \to 0^+} f(x) = 2.$$

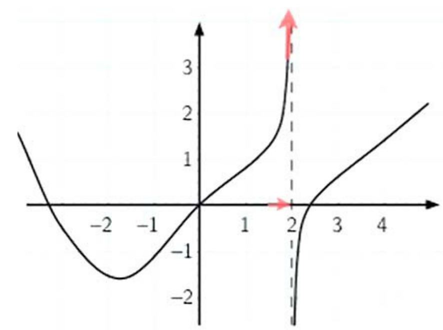
Important fact: $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist and are equal.
If it happens, the common value of the one-sided limits is $\lim_{x\to a} f(x)$.

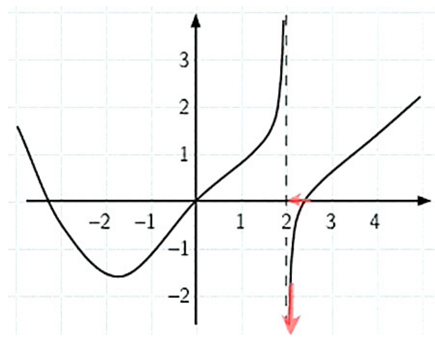
Another example. Suppose, that $f(x)$ has the graph shown below.



Here the interesting behavior of f is in the vicinity of $x = 2$. Notice, that $f(2)$ is undefined and the line $x = 2$ is a vertical asymptote. If $x \approx 2$ and $x < 2$, then $f(x)$ is large and positive. But if $x \approx 2$ and $x > 2$, then $f(x)$ is large and negative. Therefore, the limit of $f(x)$ as x approaches 2 *does not exist*. In fact, *neither of the one sided limits exist*. However, we can describe the nature of the vertical asymptote writing:

$$\lim_{x \to 2^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to 2^+} f(x) = -\infty.$$

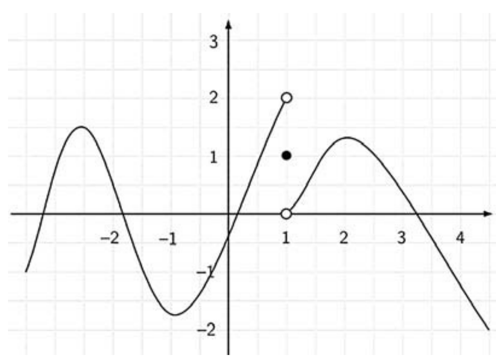




2.2 Examples

Example 4 *Given the function $f(x)$ whose graph is below, determine the following:*

a) $f(1)$ *b)* $\lim_{x\to 1^-} f(x)$ *c)* $\lim_{x\to 1^+} f(x)$ *d)* $\lim_{x\to 1} f(x)$



Solution:

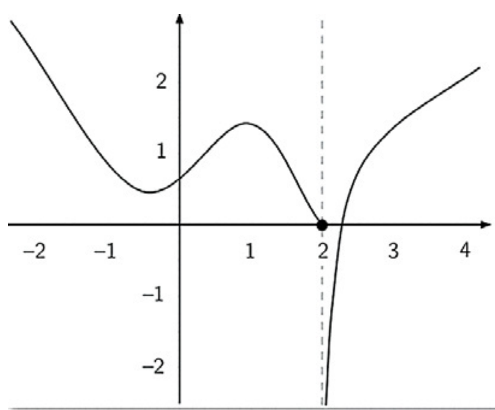
a) $f(1) = 1$ b) $\lim_{x\to 1^-} f(x) = 2$

c) $\lim_{x\to 1^+} f(x) = 0$ d) $\lim_{x\to 1} f(x)$ does not exist.

□

Example 5 *Given the function f whose graph is below, determine the following:*

a) $f(2)$ *b)* $\lim_{x\to 2^-} f(x)$ *c)* $\lim_{x\to 2^+} f(x)$ *d)* $\lim_{x\to 2} f(x)$



Solution:

a) $f(2) = 0$ b) $\lim_{x\to 2^-} f(x) = 0$

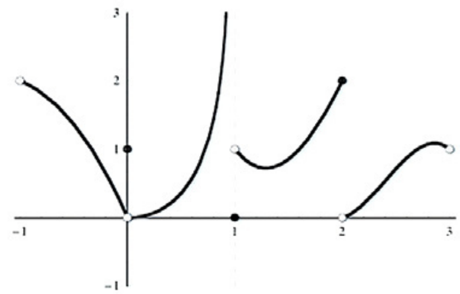
c) $\lim_{x\to 2^+} f(x) = -\infty$ d) $\lim_{x\to 2} f(x)$ does not exist.

□

Example 6 *Sketch the graph of a function f, defined for $-1 < x < 3$, for which the following are true:*

$\lim_{x\to -1^+} f(x) = 2$ $f(0) = 1$
$\lim_{x\to -0} f(x) = 0$ $f(1) = 0$
$\lim_{x\to -1^-} f(x) = \infty$ $\lim_{x\to -1^+} f(x) = 1$
$f(2) = 2$ $\lim_{x\to -2^+} f(x) = 0$
$\lim_{x\to -3^-} f(x) = 1$

Solution: Here is a possible answer:

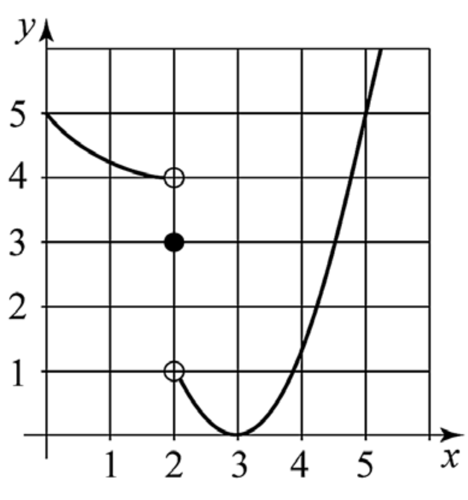


There are many other ways this graph could been drawn. Other possibilities need only indicate the correct values at, and limiting behavior near $x = -1$, 0, 1, 2, and 3.

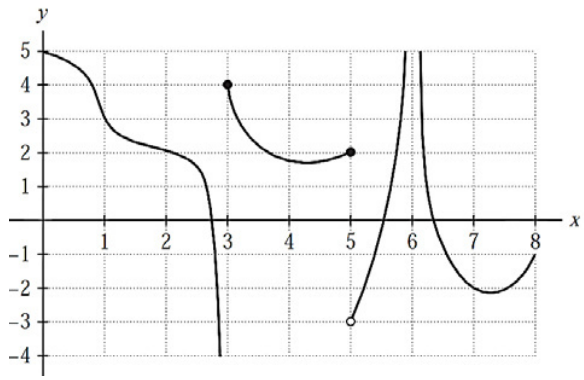
2.3 Exercises

Exercise 2.1 *Refer to the accompanying figure and determine the following:*

a) $\lim_{x\to 2^-} g(x)$ *b)* $\lim_{x\to 2} g(x)$ *c)* $\lim_{x\to 2^+} g(x)$ *d)* $\lim_{x\to 5} g(x)$

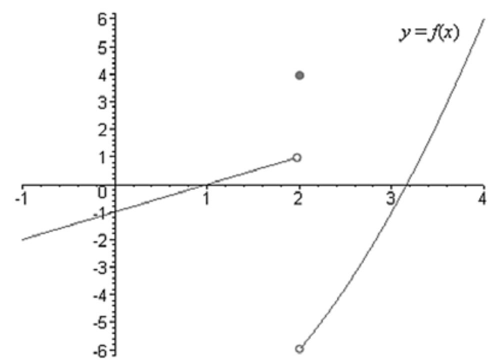


Exercise 2.2 *Determine the one-sided limits of the function $f(x)$ in figure below, at the points $c = 1, 3, 5, 6$.*



Exercise 2.3 *Use the graph of the function $f(x)$ to find the following*

a) $\lim_{x\to 2^-} f(x)$ *b)* $\lim_{x\to 2} f(x)$ *c)* $\lim_{x\to 2^+} f(x)$ *d)* $f(2)$



3

Calculation of limits (Exercises)

3.1 Some basic, general facts about limits

Some basic, general facts about limits are helpful in the calculation of specific limits.

Assume, that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ each exists.
Then the following are true:

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
3. $\lim_{x\to a}(f(x)g(x)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$
4. If $\lim_{x\to a} g(x) \neq 0$ then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
5. If $\lim_{x\to a} f(x) \neq 0$ and $\lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ does not exist
6. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ may or may not exist

While these are stated in terms of the limit as x approaches a, they are also true for one-sided limits as x approaches a.

3.2 Practice Problems

We have the following limits regarding two specific simple functions

$$\boxed{\lim_{x\to a} c = c \quad \text{and} \quad \lim_{x\to a} x = a}$$

Example 7 (a typical polynomial)

$$\begin{aligned}
\lim_{x\to 2}\left(2x^3+3x+5\right) &= \lim_{x\to 2} 2x^3 + \lim_{x\to 2} 3x + \lim_{x\to 2} 5 \\
&= 2\lim_{x\to 2} x^3 + 3\lim_{x\to 2} x + \lim_{x\to 2} 5 \\
&= 2\left(\lim_{x\to 2} x\right)^3 + 3\lim_{x\to 2} x + \lim_{x\to 2} 5 \\
&= 2\,(2)^3 + 3\cdot 2 + 5 \\
&= 27.
\end{aligned}$$

Notice that the result is simply the value of the polynomial at $x = 2$. In fact, this always happens with polynomials, that is,

$$\boxed{\begin{array}{c}\textit{If } p(x) \textit{ is a polynomial and } a \textit{ is any real number, then} \\ \lim_{x\to a} p(x) = p(a).\end{array}}$$

.□

Example 8 *(A rational function whose dominator does not approach zero).*

$$\begin{aligned}
\lim_{x\to 3}\frac{2x^2-x+2}{x^2+1} &= \frac{\lim_{x\to 3}\left(2x^2-x+2\right)}{\lim_{x\to 3}\left(x^2+1\right)} \\
&= \frac{2\,(3)^2-(3)+2}{(3)^2+1} \\
& \frac{17}{10}.
\end{aligned}$$

Again notice, that the result is simply the value of the function at $x = 3$. In fact, this always happens with rational functions, provided that the denominator isn't zero,

$$\boxed{\begin{array}{c}\textit{If } p(x) \textit{ and } q(x) \textit{ are polynomials, and } q(a)\neq 0, \textit{ then} \\ \lim_{x\to a}\dfrac{p(x)}{q(x)} = \dfrac{p(a)}{q(a)}.\end{array}}$$

□

Example 9 *(A rational function whose dominator approaches zero while its numerator does not).*

$$\lim_{x\to 1}\frac{2x^2-x+2}{x^3-1} \quad \textit{does not exist,}$$

because $\lim_{x\to 1}\left(x^3-1\right) = 0$ *while* $\lim_{x\to 1}\left(2x^2-x+2\right) = 3$.

Can we say something about one-sided limits?

$$\lim_{x\to 1^-} \frac{2x^2 - x + 2}{x^3 - 1} = -\infty$$

as the nominator is close to 3 *near* 1*, but the denominator* ≈ 0 *and* < 0. *Similarly,*

$$\lim_{x\to 1^+} \frac{2x^2 - x + 2}{x^3 - 1} = +\infty.$$

□

Recall the last of the six facts with which we began:

6. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ may or may not exist.

It is the situation that is the most interesting and, in fact, is **the main reason we are discussing limits at all** !

Computation of limits in this "$\left(\dfrac{0}{0}\right)$" case often involves use of the following additional fact about limits:

7. If $f(x) = \varphi(x)$ for all x near but not equal to a, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} \varphi(x).$$

Here, think of $\varphi(x)$ as a "simplified" version of $f(x)$ whose limit as $x \to a$ is easy to determine, such as polynomial or a rational function whose numerator and denominator do not both approach 0.

Example 10 *Find*

$$\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1}.$$

Solution:

$$\begin{aligned} \frac{x^3 - 1}{x^2 - 1} &= \frac{(x-1)\left(x + x^2 + 1\right)}{(x-1)(x+1)} \\ &= \frac{\left(x + x^2 + 1\right)}{(x+1)} \quad \text{for} \quad x \neq 1 \end{aligned}$$

So,

$$\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x\to 1} \frac{\left(x + x^2 + 1\right)}{(x+1)} = \frac{1^2 + 1 + 1}{1 + 1} = \frac{3}{2}.$$

The graph of the function $\frac{x^3-1}{x^2-1}$ is presented in Figure 3.1. Notice the "hole" in it at $(1, 3/2)$. □

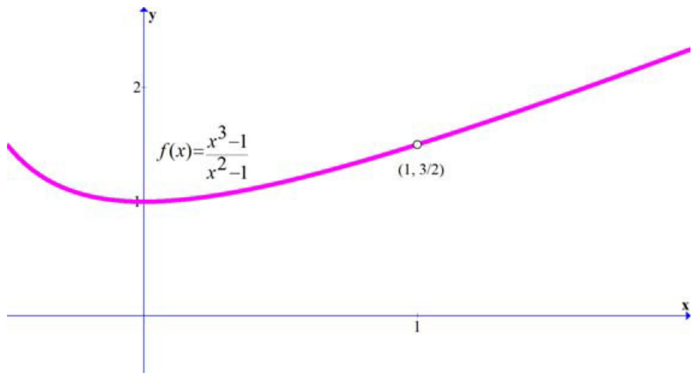


Fig. 3.1. The graph of $f(x) = \frac{x^3-1}{x^2-1}$ for x close to 1.

Example 11 *Find* $\lim_{x\to 0} \dfrac{(x+3)^2-9}{x}$.

Solution:

$$\begin{aligned} \frac{(x+3)^2-9}{x} &= \frac{x^2+6x}{x} \\ &= x+6 \quad \text{for} \quad x \neq 0 \end{aligned}$$

So,

$$\lim_{x\to 0} \frac{(x+3)^2-9}{x} = \lim_{x\to 0} (x+6) = 6.$$

The graph of the function $\dfrac{(x+3)^2-9}{x}$ is presented in Figure 3.2. Notice the "hole" in it at $(0,6)$. $\square$

Example 12 *Find* $\lim_{x\to 2} \dfrac{1/2-1/x}{x-2}$.

Solution:

$$\begin{aligned} \frac{1/2-1/x}{x-2} &= \frac{x-2}{2x(x-2)} \\ &= \frac{1}{2x} \quad \text{for} \quad x \neq 0. \end{aligned}$$

Hence,

$$\lim_{x\to 2} \frac{1/2-1/x}{x-2} = \lim_{x\to 2} \frac{1}{2x} = \frac{1}{4}.$$

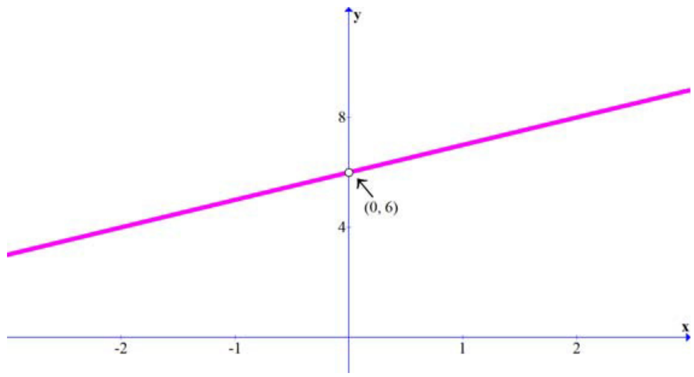


Fig. 3.2. The graph of the function $\dfrac{(x+3)^2-9}{x}$ in the vicinity of $x=0$.

□

Example 13 *Find* $\lim_{x\to 2} \dfrac{x^2-x-2}{x^2-4x+4}$

$$\begin{aligned}\frac{x^2-x-2}{x^2-4x+4} &= \frac{(x-2)(x+1)}{(x-2)^2}\\ &= \frac{x+1}{x-2} \quad \textit{for} \quad x \neq 2\end{aligned}$$

Hence

$$\lim_{x\to 2} \frac{x^2-x-2}{x^2-4x+4} \quad \textit{does not exist.}$$

However,

$$\lim_{x\to 2^-} \frac{x^2-x-2}{x^2-4x+4} = -\infty$$

and

$$\lim_{x\to 2^+} \frac{x^2-x-2}{x^2-4x+4} = \infty$$

The graph of the function $\dfrac{x^2-x-2}{x^2-4x+4}$ *is presented in Figure 3.3. It has vertical asymptote at* $x=2$.

□

For the next example, we will need an additional continuity property

$$\lim_{x\to a} \sqrt{x} = \sqrt{a} \quad \text{for all } a>0, \quad \text{and} \quad \lim_{x\to 0^+} \sqrt{x} = 0.$$

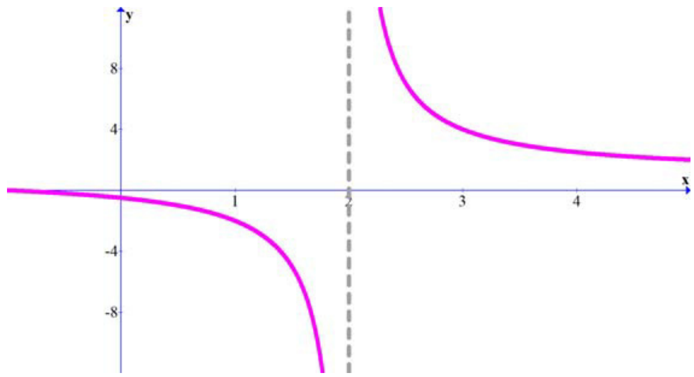


Fig. 3.3. The graph of the function

Example 14 *Find* $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$.

Solution:

$$\begin{aligned}\frac{\sqrt{x}-1}{x-1} &= \frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}\\ &= \frac{1}{\sqrt{x}+1} \quad \text{for} \quad x \neq 1.\end{aligned}$$

So,

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

□

3.3 Some Trigonometric Limits

3.3.1 Continuity of sine and cosine

For all real numbers a, $\lim_{x\to a}\sin x = \sin a$ and $\lim_{x\to a}\cos x = \cos a$

Exercise 3.1 *Find* $\lim_{x\to 0}\dfrac{\sin x + 2\cos x + \cos^2 x}{\sin x + \cos 2x}$.

Solution: As

$$\sin 0 + \cos 2\cdot 0 = 1 \neq 0$$

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x+2\cos x+\cos^2 x}{\sin x+\cos 2x} &= \frac{\lim_{x\to 0}\left(\sin x+2\cos x+\cos^2 x\right)}{\lim_{x\to 0}\left(\sin x+\cos 2x\right)}\\ &= \frac{\sin 0+2\cos 0+\cos^2 0}{\sin 0+\cos 2\cdot 0}\\ &= 3\end{aligned}$$

□

3.3.2 Tangent

If a is not an odd multiple of $\pi/2$, then $\cos a \neq 0$ and so

$$\lim_{x\to a}\tan x=\lim_{x\to a}\frac{\sin x}{\cos x}=\frac{\sin a}{\cos a}=\tan a.$$

Suppose that a is an odd multiple of $\pi/2$. Then $\sin a \neq 0$ and $\cos a = 0$ so $\lim_{x\to a}\tan x$ does not exist.
If $x \approx a$ and $x < a$, then $\sin x$ and $\cos x$ have the same sign; so $\tan x > 0$. Therefore

$$\lim_{x\to a^-}\tan x=\lim_{x\to a^-}\frac{\sin x}{\cos x}=\infty.$$

If $x \approx a$ and $x > a$, then $\sin x$ and $\cos x$ have opposite sign; so $\tan x < 0$. Therefore

$$\lim_{x\to a^+}\tan x=\lim_{x\to a^+}\frac{\sin x}{\cos x}=-\infty.$$

3.3.3 Cotangent

If a is not a multiple of π, then $\sin a \neq 0$ and so

$$\lim_{x\to a}\cot x=\lim_{x\to a}\frac{\cos x}{\sin x}=\frac{\cos a}{\sin a}=\cot a.$$

If a is a multiple of π, then $\lim_{x\to a}\cot x$ does not exist.

Note that

$$\cot x=\tan\left(\frac{\pi}{2}-x\right)=-\tan\left(x-\frac{\pi}{2}\right).$$

So, if a is a multiple of π, then

$$\lim_{x\to a^-}\cot x=-\lim_{x\to a^-}\tan\left(x-\frac{\pi}{2}\right)=-\lim_{x\to\left(a-\frac{\pi}{2}\right)^-}\tan x=-\infty,$$

and similarly

$$\lim_{x\to a^+}\cot x=-\lim_{x\to a^+}\tan\left(x-\frac{\pi}{2}\right)=-\lim_{x\to\left(a-\frac{\pi}{2}\right)^+}\tan x=+\infty.$$

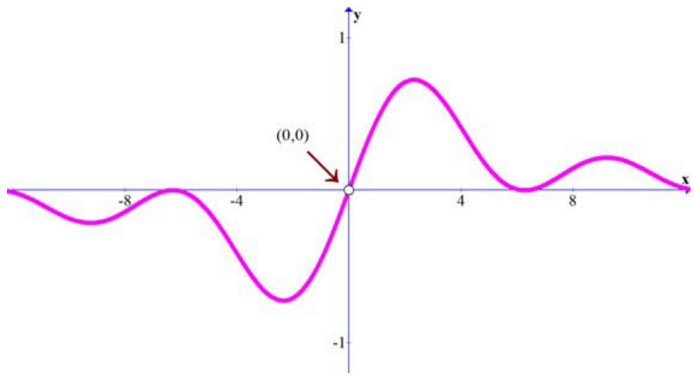


Fig. 3.4. The graoh of $f(x) = \dfrac{1-\cos x}{x}$.

3.3.4 Typical Examples

Example 15 *Find* $\lim_{x\to 0} \dfrac{1-\cos x}{x}$ *(see Figure 3.4)*

Solution:

$$
\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} &= \lim_{x\to 0} \frac{1-\cos^2 x}{x} \cdot \frac{1}{1+\cos x} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{x} \cdot \frac{1}{1+\cos x} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} \\
&= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x} \\
&= 1 \cdot \frac{0}{2} = 0.
\end{aligned}
$$

□

Example 16 *Find* $\lim_{x\to 0} \dfrac{\sin^2(\pi x)}{2x^2}$.

Solution:

$$
\begin{aligned}
\lim_{x\to 0} \frac{\sin^2(\pi x)}{2x^2} &= \lim_{x\to 0} \frac{\pi^2}{2} \frac{\sin(\pi x)}{\pi x} \frac{\sin(\pi x)}{\pi x} \\
&= \frac{\pi^2}{2} \lim_{x\to 0} \frac{\sin(\pi x)}{\pi x} \cdot \lim_{x\to 0} \frac{\sin(\pi x)}{\pi x} \\
&= \frac{\pi^2}{2} \cdot 1 \cdot 1 = \frac{\pi^2}{2}
\end{aligned}
$$

□

Example 17 *Find* $\lim_{x\to 0} \dfrac{2 - 3\cos x + \cos^2 x}{\sin x}$.

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{2 - 3\cos x + \cos^2 x}{\sin x} &= \lim_{x\to 0} \frac{(2-\cos x)(1-\cos x)}{\sin x} \\
&= \lim_{x\to 0} (2-\cos x) \frac{\dfrac{(1-\cos x)}{x}}{\dfrac{\sin x}{x}} \\
&= (2-1)\cdot 0/1 = 0
\end{aligned}$$

□

Example 18 *Find* $\lim_{x\to 0} \dfrac{1-\cos(3x)}{x^2}$.

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} &= \lim_{x\to 0} \frac{1-\cos(3x)}{x^2} \cdot \frac{1+\cos(3x)}{1+\cos(3x)} \\
&= \lim_{x\to 0} \frac{1-\cos^2(3x)}{x^2} \cdot \frac{1}{1+\cos(3x)} \\
& \quad \lim_{x\to 0} \frac{\sin(3x)}{x} \cdot \frac{\sin(3x)}{x} \frac{3\cdot 3}{1+\cos(3x)} \\
&= 1 \cdot 1 \cdot \frac{9}{2} \\
&= \frac{9}{2}.
\end{aligned}$$

□

3.4 The Epsilon-Delta Definition of a Limit

Let f be a function that is defined on an open interval containing a except possibly a itself, and let L be a real number. The statement

$$\lim_{x\to a} f(x) = L$$

means that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$\text{if} \quad 0 < |x-a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

Example 19 *Let* $f(x) = |x|/x$, *show that* $\lim_{x\to 0} f(x)$ *does not exist.*

Solution: If $x > 0$ then $|x|/x = x/x = 1$ and hence, to the right of the y-axis, the graph coincides with the line $y = 1$. If $x < 0$, then $|x|/x = -x/x = -1$, which means that to the left of the y-axis the graph of f coincides with the line $y = -1$. If it were true that $\lim_{x\to 0} f(x) = L$ for sone L, then the preceding remarks imply, that $-1 \le L \le 1$. If we consider any pair of horizontal lines $y = L \pm \varepsilon$, where $1 > \varepsilon > 0$ then there exist points on the graph which are not between these lines for some nonzero x in *every interval* $(-\delta, \delta)$ containing 0. It follows that the limit does not exist. □

Example 20 *For the limit given below, find the largest δ that works for the given ε*

$$\lim_{x\to 4} 2x = 8, \quad \varepsilon = 0.1$$

Solution: The definition tells us that $\lim_{x\to 4} 2x = 8$ iff for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if} \quad 0 < |x-4| < \delta \quad \text{then} \quad |2x-8| < \varepsilon.$$

Note that $|2x-8| = 2|x-4|$. Thus $|2x-8| < \varepsilon$ whenever $2|x-4| < \varepsilon$ or $|x-4| < \frac{\varepsilon}{2}$. From the work shown above, we see that the largest choice of δ that works for $\varepsilon > 0$ is $\delta = \varepsilon/2$. □

Exercise 3.2 *Give an $\varepsilon - \delta$ proof for the following limit*

$$\lim_{x\to 3} (3x+1) = 10$$

Solution: The definition tells us that $\lim_{x\to 3} (3x+1) = 10$ iff for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |(3x+1)-10| < \varepsilon.$$

Note that

$$|(3x+1)-10| = |3x-9| = 3|x-3|.$$

Thus $|(3x+1)-10| < \varepsilon$ whenever $3|x-3| < \varepsilon$ or $|x-3| < \varepsilon/3$.

Now let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$.

If $0 < |x-3| < \delta$ then

$$|(3x+1)-10| = |3x-9| = 3|x-3| < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

By the $\varepsilon - \delta$ definition of a limit, $\lim_{x\to 3} (3x+1) = 10$. □

3.5 Exercises

Exercise 3.3 *Why is it impossible to investigate* $\lim_{x\to 0}\sqrt{x}$ *by means of the Epsilon-Delta Definition of a Limit.*

Exercise 3.4 *For the limit given below, find the largest* δ *that works for the given* ε

$$\lim_{x\to 4}\frac{1}{5}x=\frac{4}{5},\quad \varepsilon=0.01$$

`Answer:` $\delta=0.05$

Exercise 3.5 *Give an* $\varepsilon-\delta$ *proof for the following limit*

$$\lim_{x\to 2}(6-4x)=-2.$$

Exercise 3.6 *Give an* $\varepsilon-\delta$ *proof for the following limit*

$$\lim_{x\to -1}(2-7x)=9.$$

4

Continuity (Exercises)

4.1 Continuous Functions

Let f be a function whose domain includes an open interval centered at $x = a$. Then f is said to be **continuous at** a if $\lim_{x\to a} f(x) = f(a)$

Example 21 *Let's $f(x)$ be the function with the graph presented in Figure 4.1 Then*

- *f is continuous everywhere* except *at* $x = -1, \quad 0.$ *and* 2.
- *f is not continuous at -1 because $\lim_{x\to -1} f(x) = 1$ while $f(-1) = 2$.*
- *f is not continuous at 0 because $\lim_{x\to 0} f(x)$ does not exist.*
- *f is not continuous at 2 because $f(2)$ is not defined.*

4.1.1 Three types of "simple" discontinuities

Example 22 *Let's $f(x)$ be the function with the graph presented in Figure 4.2 Then it has*

- *"Jump" discontinuity at $x = -1$ (one-sided limits exist but are different).*
- *"Infinite" discontinuity at $x = 0$ (An infinite one-sided limit).*
- *"Removable" discontinuity (the limit exists but does not equal the value of function)*

4.1.2 Classical continuous functions

- Polynomials are continuous everywhere.
- Rational functions are continuous whenever they are defined.
- $\sin x$ and $\cos x$ are continuous everywhere.

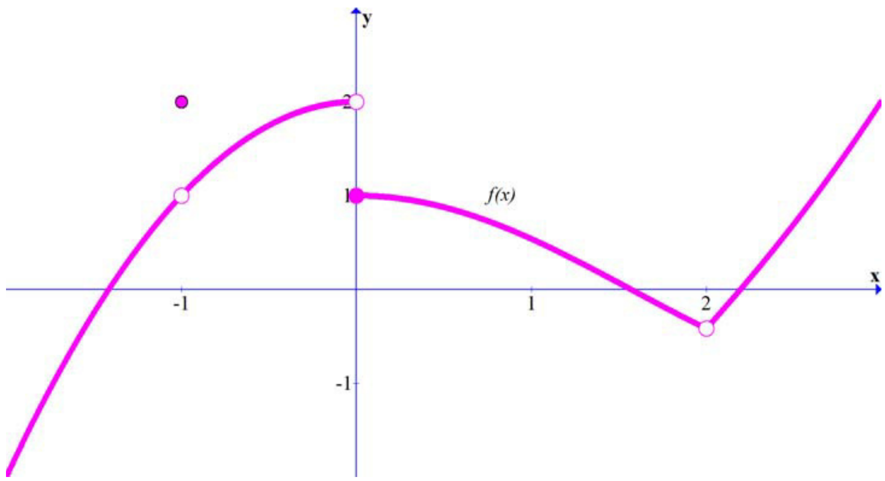


Fig. 4.1. The graph of an exemplary function $f(x)$

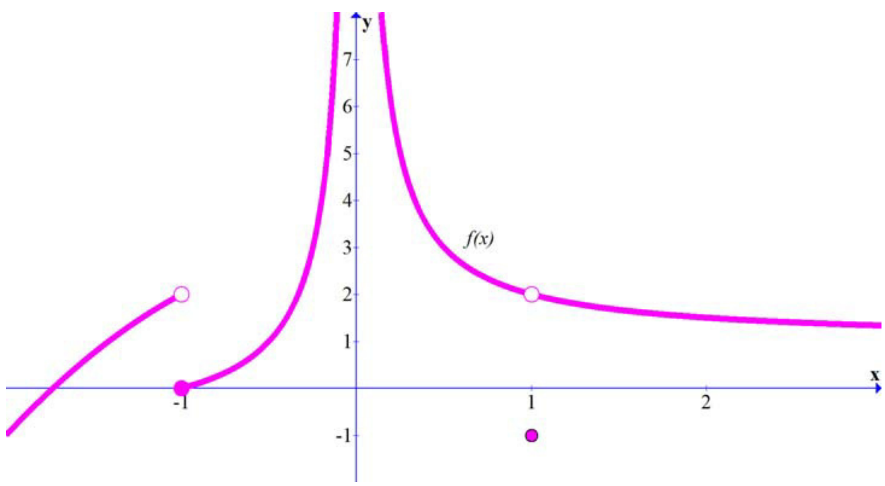
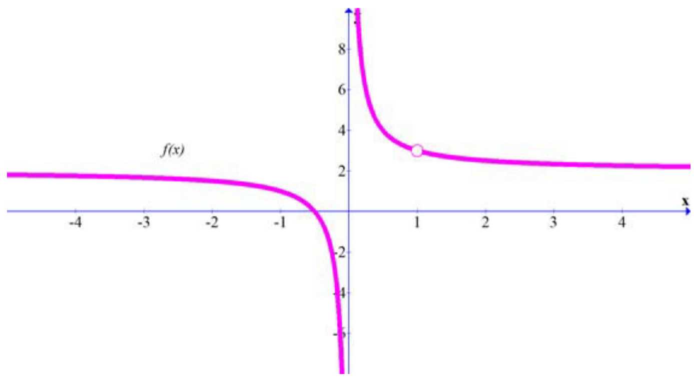


Fig. 4.2. Three types of "simple" discontinuities

- $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are continuous whenever they are defined.

Example 23 *The rational function*

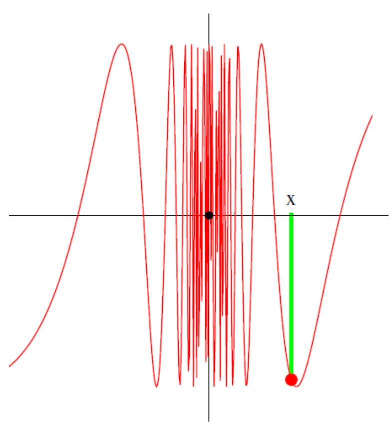
$$f(x) = \frac{(x-1)(2x+1)}{x(x-1)}$$



has only two discontinuity points at

1. $x = 0$ *("Infinite" discontinuity).*
2. $x = 1$ *("Removable" discontinuity).*

Example 24 *The function f(x)* $= \sin(\pi/x)$ *is continuous everywhere except at* $x = 0$*. It is a prototype of a function which is not continuous due to **oscillation**. We can approach* $x = 0$ *in ways that* $f(x_n) = 1$ *and such that* $f(z_n) = -1$*. Just chose* $x_n = 2/(4k+1)$ *or* $z_n = 2/(4k-1)$*.*



4.1.3 Several properties of continuous functions

Directly from the properties of the limits it follows that

If f and g are both continuous at a, then

1. $f+g$ and $f-g$ are continuous at a,
2. cf is continuous at a,
3. fg is continuous at a,
4. if $g(a) \neq 0$, then f/g is continuous at a.

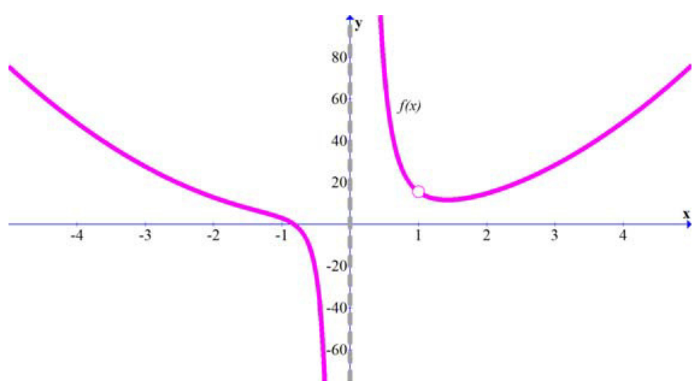
Compositions

5. if g is continuous at a and f is continuous at $f(a)$ then $f \circ g$ is continuous at a.

Example 25 *Let*

$$f(x) = 3x^2 + \frac{x^2-1}{(x-1)(x-\sin x)}.$$

At what values of x is f ***not*** *continuous? Of what type is each of its discontinuities?*



Solution: First observe, that $3x^2$, x^2-1, and $(x-1)(x-\sin x)$ are each continuous everywhere, since they involve only polynomials and sine. Therefore, f is continuous whenever $(x-1)(x-\sin x) \neq 0$, that is, everywhere except at $x=1$ and $x=0$.

To determine the type of discontinuity at $x=1$, we will examine $\lim_{x\to 1} f(x)$

$$f(x) = 3x^2 + \frac{x^2-1}{(x-1)(x-\sin x)} = 3x^2 + \frac{x+1}{(x-\sin x)} \quad \text{for} \quad x \neq 1.$$

So, the limit exists:

$$\lim_{x\to 1} f(x) = 3\cdot 1 + \frac{1+1}{(1-\sin 1)} \approx 15.616.$$

Therefore, the discontinuity at $x = 1$ is **removable,** corresponding to a "hole" in the graph of f.

To determine the type of discontinuity at $x = 0$, we first notice that

$$\lim_{x\to 0} f(x) \quad \text{does not exist,}$$

and in fact

$$\lim_{x\to 0^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to 0^+} f(x) = \infty.$$

Therefore, f has an infinite discontinuity at $x = 0$, corresponding to a vertical asymptote. □

Example 26 *Find numbers a and b such that the following function is continuous everywhere*

$$f(x) = \begin{cases} ax & \text{if} \quad x \le 1 \\ x^2 + a - b & \text{if} \quad -1 \le x < 1 \\ bx & \text{if} \quad 1 \le x \end{cases}$$

Solution: Since the "parts" of f are polynomials, we only need to choose a and b so that f is continuous at $x = -1$ and 1. we have

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$$

$$\lim_{x\to -1^-} ax = \lim_{x\to -1^+} x^2 + a - b = f(-1)$$

$$-a = 1 + a - b$$

$$-2a + b = 1$$

and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\lim_{x\to 1^-} x^2 + a - b = \lim_{x\to 1^+} bx$$

$$1 + a - b = b$$

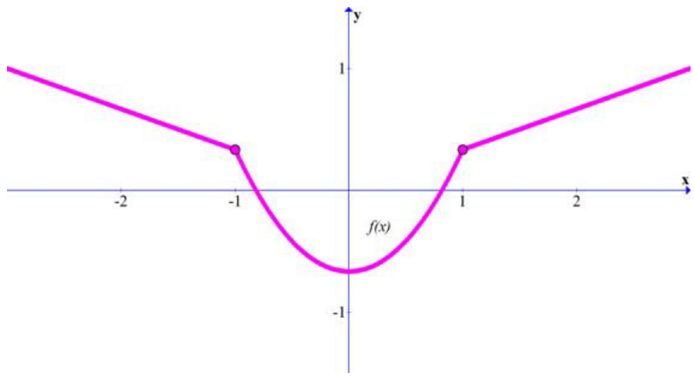


Fig. 4.3. Continuous function from Example 26

$$a - 2b = -1.$$

Solving the system of two linear equations

$$\begin{cases} -2a + b & = & 1 \\ \\ a - 2b & = & -1 \end{cases}$$

which is equivalent to

$$\begin{array}{rcl} -3b & = & -1 \\ a - \frac{2}{3} & = & -1 \end{array}$$

we get

$$a = -\frac{1}{3}, \qquad b = \frac{1}{3}.$$

(see Figure 4.3) $\square$

Example 27 (Composition) *Let*

$$f(x) = \frac{x}{|x|} = \begin{cases} -1 & \textit{if} \quad x < 0 \\ \\ 1 & \textit{if} \quad x \geq 0 \end{cases}$$

If we compose f *with* $\sin x$ *the result*

$$f(x) = \frac{\sin x}{|\sin x|}$$

will be continuous, whenever $\sin x \neq 0$, *i.e. except multipliers of* π*. At multipliers of* π*, it is undefined and has a jump discontinuity (see Figure 4.4).*

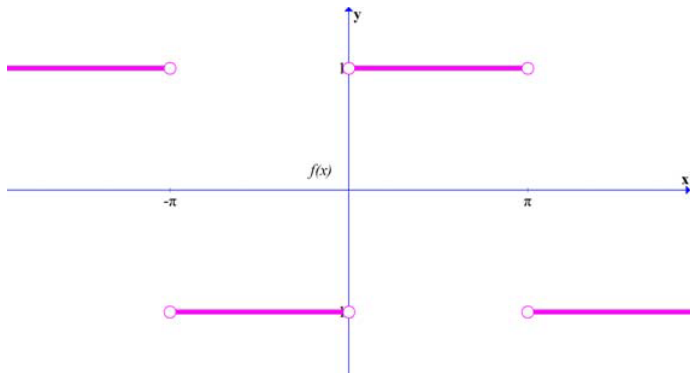


Fig. 4.4. Function $f(x) = \frac{\sin x}{|\sin x|}$.

4.1.4 One sided continuity

If the domain of f includes an interval whose right endpoint is a, then f is *left continuous* at a if

$$\lim_{x\to a^-} f(x) = f(a).$$

If the domain of f includes an interval whose left endpoint is a, then f is *right continuous* at a if

$$\lim_{x\to a^+} f(x) = f(a).$$

Function $f(x)$ is continuous at a if and only if f is both left continuous and right continuous at a.

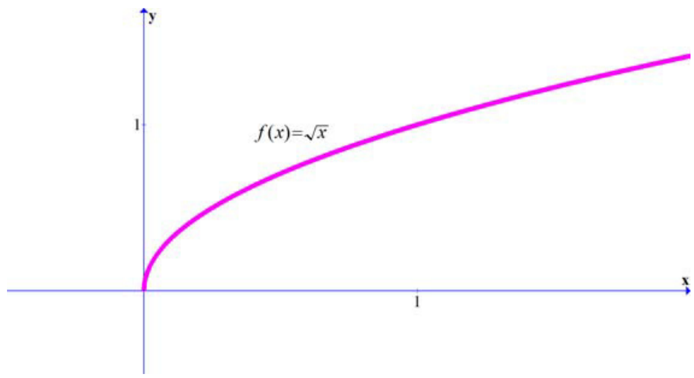
Example 28

$$\lim_{x\to a^+} \sqrt{x} = \sqrt{a} \qquad \textit{for all} \quad a > 0,$$

and

$$\lim_{x\to 0^+} \sqrt{x} = 0.$$

So square root function is continuous at every positive number and right continuous at 0.



4.1.5 Continuity on an interval

Let I be an interval i.e., a set of one of the following forms

$$(a, b), \quad [a, b], \quad [a, b), \quad (a, b], \quad (a, \infty), \quad (-\infty, b), \quad (-\infty, b], \quad \text{or} \quad (-\infty, \infty).$$

A function f whose domain includes I is said to be continuous on I, if for every number c in I

- f is continuous at c if c is not an endpoint of I.
- f is right continuous at c if c is a left endpoint of I.
- f is left continuous at c if c is a right endpoint of I.

If the domain of f is an interval, and if f is continuous on that interval, then we may simply say that f is a **continuous function**.

Example 29 **a)** $f(x) = \frac{1}{x}$ *is continuous on* $(0, \infty)$ *and on* $(-\infty, 0)$.

b) $f(x) = \sqrt{x}$ *is continuous on* $[0, \infty)$.

c) $f(x) = \sqrt{1 - x^2}$ *is continuous on* $[-1, 1)$.

4.2 Exercises

Exercise 4.1 *Let*

$$f(x) = \begin{cases} 2x + 1 & \textit{if} \quad x \leq 1 \\ x^2 - 1 & \textit{if} \quad x > 1. \end{cases}$$

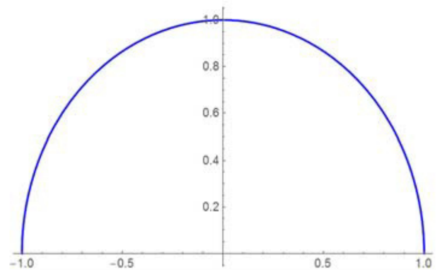


Fig. 4.5. Function $f(x) = \sqrt{1-x^2}$.

Is f continuous at $x = 1$? If not, what type of discontinuity does f have at $x = 1$?

`Answer:` The function f is discontinuous at $x = 1$, and has jump discontinuity there.

Exercise 4.2 *Let*

$$f(x) = \begin{cases} 2x+1 & \text{if} \quad x \leq 1 \\ 0 & \text{if} \quad x = 1 \\ x^2+2 & \text{if} \quad x > 1. \end{cases}$$

Is f continuous at $x = 1$? If not, what type of discontinuity does f have at $x = 1$?

`Answer:` The function f is discontinuous at $x = 1$, and has removable discontinuity there.

Exercise 4.3 *Let*

$$f(x) = \begin{cases} 2x+1 & \text{if} \quad x \leq 1 \\ 0 & \text{if} \quad x = 1 \\ \frac{1}{x-1} & \text{if} \quad x > 1. \end{cases}$$

Is f continuous at $x = 1$? If not, what type of discontinuity does f have at $x = 1$?

`Answer:` The function f is discontinuous at $x = 1$, and has infinite discontinuity there.

Exercise 4.4 *Let*

$$f(x) = \begin{cases} A^2x^2 & \text{if} \quad x \leq 2 \\ \\ (1-A)x & \text{if} \quad x > 2. \end{cases}$$

For what values of A is f continuous at $x = 2$?

`Answer:` $A = 1/2$ or $A = -1$.

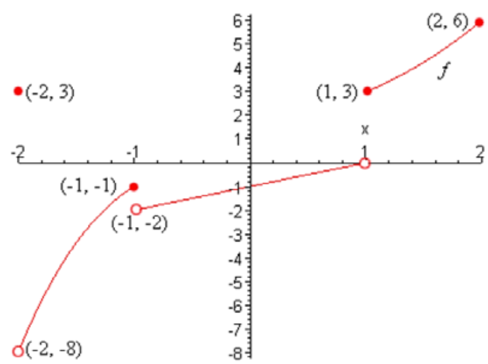
Exercise 4.5 *Let*

$$f(x) = \frac{\sqrt{x+4} - 3}{\sqrt{x-5}}.$$

If possible define f at $x = 5$ so that f is continuous at $x = 5$.

`Answer:` $f(5) = 0$.

Exercise 4.6 *The graph of the function f is shown in the figure below. Determine the intervals on which f is continuous.*



`Answer:` $(-2, -1]$, $(-1, 1)$, $[1, 2]$

Exercise 4.7 *Let*

$$f(x) = \begin{cases} x + 1 & \textit{if} \quad x \leq 1 \\ \\ \frac{1}{x-1} & \textit{if} \quad x > 1. \end{cases}$$

Determine the intervals on which f is continuous.

`Answer:` $(-\infty, 1)$, $(1, \infty)$

Exercise 4.8 *Let*

$$f(x) = \begin{cases} -2 & \textit{if} \quad x \leq -1 \\ |x| & \textit{if} \quad -1 < x < 1 \\ \frac{1}{x-1} & \textit{if} \quad x \geq 1. \end{cases}$$

Determine the intervals on which f is continuous.

`Answer:` $(-\infty, 1]$, $(1, \infty)$.

Exercise 4.9 *Let*

$$f(x) = \begin{cases} -2 & \text{if} \quad x \leq -1 \\ |x| & \text{if} \quad -1 < x < 1 \\ \frac{1}{x-1} & \text{if} \quad x \geq 1. \end{cases}$$

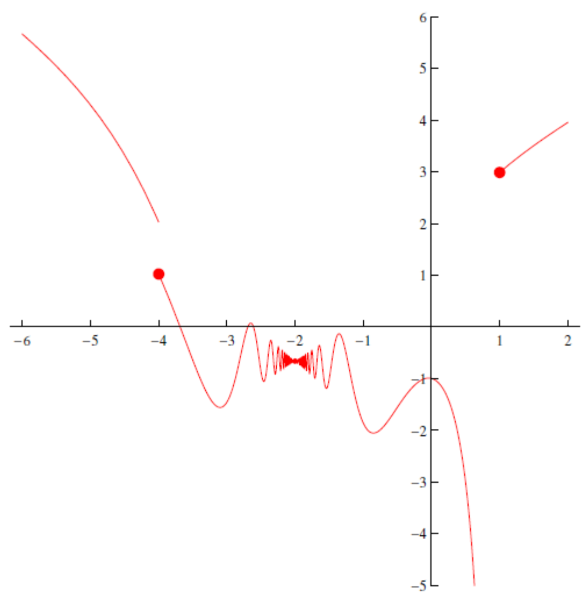
a) *Sketch the graph of f.*

b) *At which points is f discontinuous?*

c) *For each point of discontinuity found in b) determine whether f is continuous from the right, from the left, or neither.*

d) *Classify each discontinuity found in b) as being a removable discontinuity, jump discontinuity, or an infinite discontinuity.*

Exercise 4.10 *On which intervals is the following function continuous?*

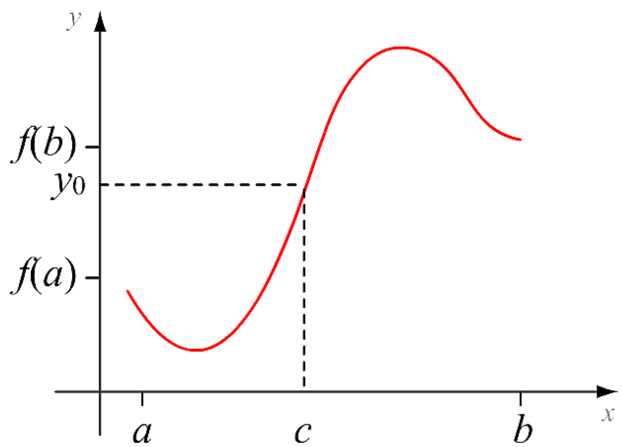


5

Benefits of Continuity (Exercises)

5.1 The Intermediate Value Theorem

Theorem 30 (The Intermediate Value Theorem) *Suppose $f(x)$ is continuous on the interval $[a, b]$ and let y_0 be a number between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) such that $f(c) = y_0$.*



Example 31 *Use the Intermediate Value Theorem to show that there is a solution of the given equation in the indicated interval*

$$x^5 - 2x^4 - x - 3 = 0, \quad [2.3]$$

Solution:
Scratch Work:
To show that the equation $x^5 - 2x^4 - x - 3 = 0$ has a solution on the interval [2.3], follow the steps below:

1. Let $f(x) = x^5 - 2x^4 - x - 3$. Observe, that f is continuous on [2.3].
2. Find $f(2)$ and $f(3)$ to see that $f(2) \cdot f(3) < 0$ (opposite signs)

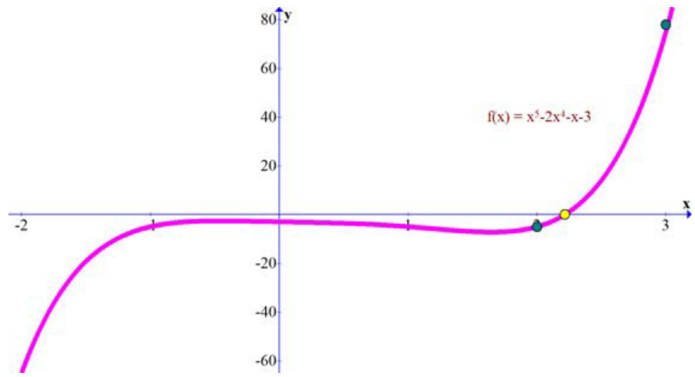


Fig. 5.1. The equation $x^5 - 2x^4 - x - 3 = 0$, has a solution in [2.3]

3. Note that $K = 0$ is the number between $f(2)$ and $f(3)$.
4. f is continuous on $[2, 3]$ and 0 is a number between $f(2)$ and $f(3)$. So, use the Intermediate Value Theorem to conclude that there is at least one number c in the interval $(2, 3)$ such that $f(c) = 0$.

Let $f(x) = x^5 - 2x^4 - x - 3$. Since f is a polynomial function, it is continuous on the closed interval $[2, 3]$. Now, $f(2) = -5$ and $f(3) = 7$. Thus $f(2) < 0 < f(3)$. Hence , $K = 0$ is a number between $f(2)$ and $f(3)$. By the Intermediate Value Theorem, there is at least one number c in the interval $(2, 3)$ such that $f(c) = 0$. Therefore, the equation $x^5 - 2x^4 - x - 3 = 0$, has a solution in the interval [2.3] (see Figure 5.1). □

Example 32 *Let*

$$f(x) = x^4 - 3x^2 + 6x$$

show that there is a number c such that $f(c) = 1$.

Solution: Let $f(x) = x^4 - 3x^2 + 6x$. Since f is a polynomial function, it is continuous on the closed interval $[-3, -2]$. Now, $f(-3) = 36$ and $f(-2) = -8$. Thus, $f(-2) < 1 < f(-3)$. Hence, $K = 1$ is a number between $f(-3)$ and $f(-2)$. By the Intermediate Value Theorem, **there is at least one number** c in the interval $(-3, -2)$ such that $f(c) = 1$. (see Figure 5.2) □

Example 33 *Solve the given inequality for x*

$$\frac{x^2 - 3x}{x + 1} < 0.$$

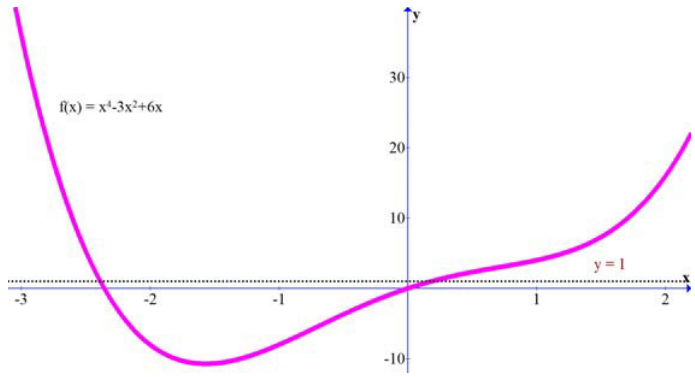


Fig. 5.2. The graph of the function $f(x) = x^4 - 3x^2 + 6x$.

Solution: Let

$$R(x) = \frac{x^2 - 3x}{x + 1}.$$

Since

$$R(x) = \frac{x(x-3)}{x+1},$$

we see that $R(x) = 0$ for $x = 0$ and $x = 3$. Also, $R(x)$ is not defined at $x = -1$.

The numbers $-1, 0$ and 3 partition the real line into 4 subintervals

$$(-\infty, -1), \quad (-1, 0), \quad (0, 3), \quad \text{and} \quad (3, \infty).$$

The rational function $R(x) = \frac{x(x-3)}{x+1}$ is continuous and nonzero on each of these intervals. Therefore, either $R(x) < 0$ or $R(x) > 0$ on each of the subintervals.

To determine the sign of $R(x)$ on each interval, choose a number α in each subinterval and evaluate $R(\alpha)$.

- Choose $\alpha = -2$ on the subinterval $(-\infty, -1)$,

  $$R(-2) = -10,$$

  so $R(x) < 0$ on the subinterval $(-\infty, -1)$.

- Choose $\alpha = -\frac{1}{2}$ on the subinterval $(-1, 0)$,

  $$R(-\frac{1}{2}) = \frac{7}{2},$$

  so $R(x) > 0$ on the subinterval $(-1, 0)$.

- Choose $\alpha = 1$ on the subinterval $(0, 3)$,

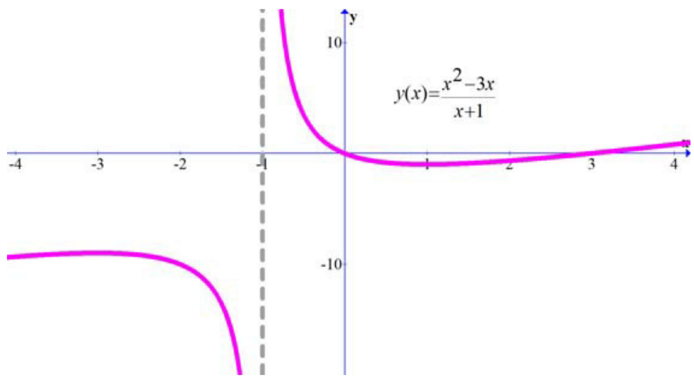
$$R(1) = -1,$$

so $R(x) < 0$ on the subinterval $(0, 3)$.

- Choose $\alpha = 4$ on the subinterval $(3, \infty)$,

$$R(4) = \frac{4}{5},$$

so $R(x) > 0$ on the subinterval $(3, \infty)$. Therefore, the solution set of the inequality $\frac{x^2-3x}{x+1} < 0$ is $(-\infty, -1) \cup (0, 3)$.



□

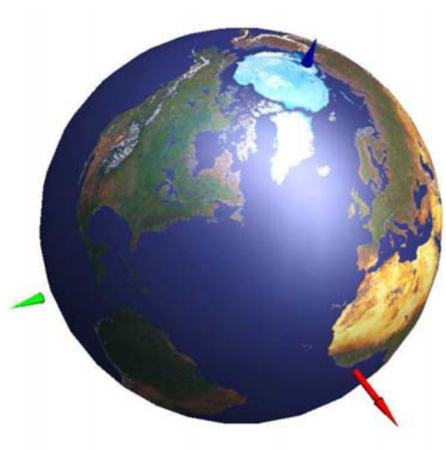
Example 34 *At* 8 : 00 *A.M. on Saturday a man begins running up the side of a mountain to his weekend campsite (see figure). On Sunday morning at* 8 : 00 *A.M. he runs back down the mountain. It takes him* 20 *minutes to run up, but only* 10 *minutes to run down. At some point on the way down, he realizes that he passed the same place at exactly the same time on Saturday. Prove that he is correct.*

Solution: `HINT`: Let $s(t)$ and $r(t)$ be the position functions for the runs up and down, and apply the Intermediate Value Theorem to the function $f(t) = s(t) - r(t)$.

Example 35 *Verify that the function* $f(x) = x^{17} - x^3 + x^5 + 5x^7 + \sin(x)$ *has a root.*

Solution: The function goes to $+\infty$ for $x \to \infty$ and to $-\infty$ for $x \to -\infty$. We have for example $f(10000) > 0$ and $f(-1000000) < 0$. The intermediate value theorem assures there is a point c where $f(c) = 0$. $\square$

Example 36 *There is a point on the earth, where temperature and pressure agrees with the temperature and pressure on the antipode.*



Solution: Lets draw a meridian through the north and south pole and let $f(x)$ be the temperature on that circle. Define $g(x) = f(x) - f(x + \pi)$. If this function is zero on the north pole, we have found our point. If not, $g(x)$ has different signs on the north and south pole. There exists therefore an x, here the temperature is the same. Now, for every meridian, we have a latitude value $l(x)$ for which the temperature works. Now define $h(x) = l(x) - l(x + \pi)$. This function is continuous. Start with meridian 0. If $h(0) = 0$ we have found our point. If not, then $h(0)$ and $h(\pi)$ take different signs. By the intermediate value theorem again, we have a root of h. At this point both temperature and pressure are the same than on the antipode. **Remark:** this argument in the second part is not yet complete. Do you see where the problem is?

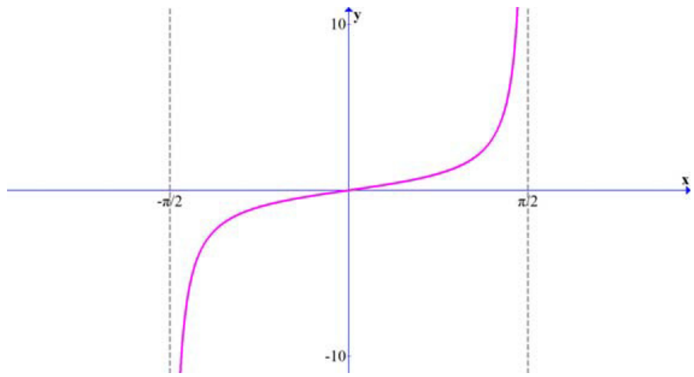


Fig. 5.3. Function $f(x)$ which is continuous on $(-\pi/2, \pi/2)$ but has no minimum value, and has no maximum value.

5.2 Boundedness; The Extreme Value Theorem

Theorem 37 (The Extreme Value Theorem) *Suppose that f is continuous on a closed bounded interval $[a, b]$. Then there exist numbers c and d in $[a, b]$ such that*

$$f(c) \leq f(x) \leq f(x) \qquad \textit{for all } x \textit{ in } [a, b].$$

Example 38 *Give an example of a function f which is continuous on an open interval (a, b), but it has no minimum value and it has no maximum value*

Solution: See Figure 5.3

Example 39 *Give an example of a function f which is defined but not continuous on an closed interval $[a, b]$, but it has no minimum value.*

Solution: Figure 5.4 presents such a function.

5.3 Exercises

Exercise 5.1 *Given that f and g are continuous on $[a, b]$ such that $f(a) > g(a)$ and $f(b) < g(b)$, prove that there is a number c in (a, b) such that $f(a) = f(b)$.*

Exercise 5.2 *Prove that the equation*

$$\frac{x^2+1}{x+3} + \frac{x^4+1}{x-4} = 0$$

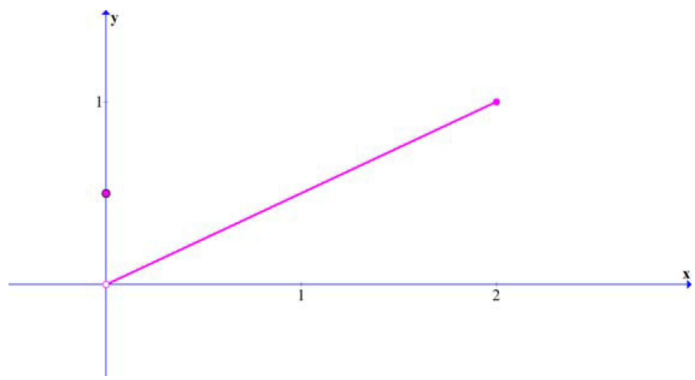


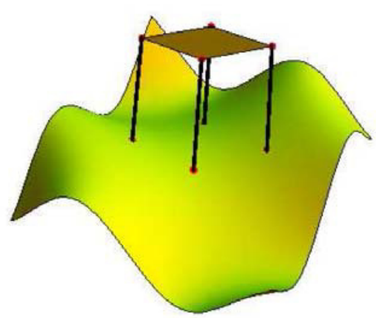
Fig. 5.4. This function has no minimum value on $[0, 2]$.

has a solution in the interval $(-3, 4)$.

Exercise 5.3 *Show, that there is a solution to the equation* $x^x = 10$.

Exercise 5.4 *Does the function* $f(x) = x + \ln|\ln|x||$ *have a root somewhere?*

Exercise 5.5 *Prove that on an arbitrary floor, a square table can be turned so that it does not wobble any more.*



Exercise 5.6 *Sketch the graph of a function* f *that is defined on the interval* $[1, 2]$ *and meets the given conditions (if possible).*

a) f *is continuous on* $[1, 2]$,

b) *The maximum value of* f *on* $[1, 2]$ *is* 3,

c) *The maximum value of* f *on* $[1, 2]$ *is* 1.

Exercise 5.7 *Sketch the graph of a function f that is defined on the interval $[1, 2]$ and meets the given conditions (if possible).*

a) *f is continuous on $(1, 2)$,*

b) *The function takes only three distinct values.*

Exercise 5.8 *Sketch the graph of a function f that is defined on the interval $[1, 2]$ and meets the given conditions (if possible).*

a) *f is continuous on $(1, 2)$,*

b) *the range of f is an unbounded interval*

Exercise 5.9 *Sketch the graph of a function f that is defined on the interval $[1, 2]$ and meets the given conditions (if possible).*

a) *f is continuous on $[1, 2]$,*

b) *the range of f is an unbounded interval.*

6

The Derivative (Exercises)

7

How to solve differentiation problems (Exercises)

8

The derivative and graphs (Exercises)

Differential calculus provides tests for locating the key features of graphs.

8.1 Practice Problems

8.1.1 Continuity and the Intermediate Value Theorem

If a continuous function on a closed interval has opposite signs at the endpoints, it must be zero at some interior point.

Example 40 *Show that the function* $f(x) = (x-1)/3x^2$ *is continuous at* $x_0 = 4$.

Solution: This is a rational function whose denominator does not vanish at $x_0 = 4$, so it is continuous by the rational function rule. □

Example 41 *Let* $g(x)$ *be the step function defined by*

$$g(x) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ \\ 1 & \text{if} \quad x > 0 \end{cases}$$

Show that g *is not continuous at* $x_0 = 0$*. Sketch.*

Solution: The graph of g is shown in Fig. 8.1. Since g approaches (in fact, equals) 0 as x approaches 0 from the left, but approaches 1 as x approaches 0 from the right, $\lim_{x \to 0} g(x)$ does not exist. Therefore, g is not continuous at $x_0 = 0$. □

We proved the following theorem: *f is differentiable at x_0, then f is continuous at x_0.* Using our knowledge of differential calculus, we can use this relationship to establish the continuity of additional functions or to confirm the continuity of functions originally determined using the laws of limits.

Example 42

a) *Show that* $f(x) = 3x^2/(x^3 - 2)$ *is continuous at* $x_0 = 1$*. Where else is it continuous?*

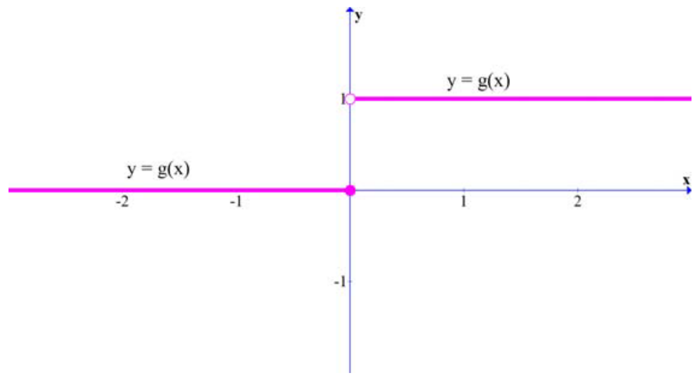


Fig. 8.1. This step function is discontinuous at $x_0 = 0$.

b) *Show that* $f(x) = \sqrt{x^2 + 2x + 1}$ *is continuous at* $x = 0$.

Solution:

a) By our rules for differentiation, we see that this function is differentiable at $x_0 = 1$; indeed, $x^3 - 2$ does not vanish at $x_0 = 1$. Thus f is also continuous at $x_0 = 1$. Similarly, f is continuous at each x, such that $x^3 - 2 \neq 0$, i.e., at each $x \neq \sqrt[3]{2}$.

b) This function is the composition of the square root function $h(u) = \sqrt{u}$ and the function $g(x) = x^2 + 2x + 1$; $f(x) = h(g(x))$. Note that $g(0) = 1 > 0$. Since g is differentiable at any x (being a polynomial), and h is differentiable at $u = l$, f is differentiable at $x = 0$ by the chain rule. Thus f is continuous at $x = 0$. $\square$

A continuous function is one whose graph never "jumps." The definition of continuity is local since continuity at each point involves values of the function only near that point. There is a corresponding global statement, called the Intermediate Value Theorem, which involves the behavior of a function over an entire interval $[a, b]$.

Example 43 *Show that there is a number* x*, such that* $x_0^5 - x_0 = 3$.

Solution: Let $f(x) = x^5 - x$. Then $f(0) = 0$ and $f(2) = 30$. Since $0 < 3 < 30$, the intermediate value theorem guarantees that there is a number x_0, in $(0, 2)$ such that $f(x_0) = 3$. (The function f is continuous on $[0, 2]$ because it is a polynomial.) $\square$

Notice that the intermediate value theorem does not tell us how to find the number x_0, but merely that it exists. (In fact there may be more than one

possible choice for x_0.) Nevertheless, by repeatedly dividing an interval into two or more parts and evaluating $f(x)$ at the dividing points, we can solve the equation $f(x_0) = c$ as accurately as we wish. This method of *bisection* is illustrated in the next example.

Example 44 (The method of bisection) *Find a solution of the equation* $x^5 - x = 3$ *in* $(0, 2)$ *to within an accuracy of* 0.1 *by repeatedly dividing intervals in half and testing each half for a root.*

Solution: In Example 43 we saw that the equation has a solution in the interval $(0, 2)$. To locate the solution more precisely, we evaluate $f(1) = 1^5 - 1 = 0$. Thus $f(1) < 3 < f(2)$, so there is a root in $(1, 2)$. Now we bisect $[1, 2]$ into $[1, 1.5]$ and $[1.5, 2]$ and repeat: $f(1.5) \backsimeq 6.09 > 3$. so there is a root in $(1, 1.5)$; $f(1.25) = 1.80 < 3$, so there is a root in $(1.25, 1.5)$; f(1.375)$\backsimeq 3.54 > 3$, so there is a root in $(1.25, 1.375)$; thus $x_0 = 1.3$ is within 0.1 of a root. Further accuracy can be obtained by means of further bisections. $\square$

8.1.2 Increasing and and Decreasing Functions

The sign of the derivative indicates whether a function is increasing or decreasing.

Example 45 *Show that* $f(x) = x^2$ *is increasing at* $x_0 = 2$.

Solution: Choose (a, b) to be, say, $(1, 3)$. If $1 < x < 2$, we have $f(x) = x^2 < 4 = x_0^2$. If $2 < x < 3$, then $f(x) = x^2 > 4 = x_0^2$. We have verified conditions of the definition, so f is increasing at $x_0 = 2$. $\square$

Example 46

a) *Is* $x^5 - x^3 - 2x^2$ *increasing or decreasing at* -2?

b) *Is* $g(s) = \sqrt{s^2 - s}$ *increasing or decreasing at* $s = 2$?

Solution:

a) Letting $f(x) = x^5 - x^3 - 2x^2$, we have $f'(x) = 5x^4 - 3x^2 - 4x$, and $f'(-2) = 5(-2), -3(-2)2 - 4(-2) = 80 - 12 + 8 = 76$, which is positive. Thus $x^5 - x^3 - 2x^2$ is increasing at -2.

b) By the chain rule, $g'(s) = \frac{1}{2}\frac{2s-1}{\sqrt{s^2-s}}$ so $g'(2) = \frac{3}{4}\sqrt{2} > 0$. Thus g is increasing at $s = 2$. $\square$

Example 47 *Let* $f(x) = x^3 - x^2 + x + 3$. *How does f change sign at* $x = -1$?

Solution: Notice that $f(-1) = 0$. Also, $f'(-1) = 3(-1)2 - 2(-1) + 1 = 6 > 0$, so f is increasing at $x = -1$. Thus f changes sign from negative to positive. $\square$

Example 48 *On what intervals is $f(x) = x^3 - 2x + 6$ increasing or decreasing?*

Solution: We consider the derivative $f'(x) = 3x^2 - 2$. This is positive when $3x^2 - 2 > 0$, i.e., when $x^2 > 2/3$, i.e., either $x > \sqrt{2/3}$ or $x < -\sqrt{2/3}$. Similarly, $f'(x) < 0$ when $x^2 < 2/3$, i.e., $-\sqrt{2/3} < x < \sqrt{2/3}$. Thus, f is increasing on the intervals $(-\infty, -\sqrt{2/3})$ and $(\sqrt{2/3}, \infty)$, and f is decreasing on $(-\sqrt{2/3}, \sqrt{2/3})$. $\square$

Example 49 *Match each of the functions in the left-hand column of Fig. 8.2 with its derivative in the right-hand column.*

Solution: Function (1) is decreasing for $x < 0$ and increasing for $x > 0$. The only functions in the right-hand column which are negative for $x < 0$ and positive for $x > 0$ are (a) and (c). We notice, further, that the derivative of function 1is not constant for $x < 0$ (the slope of the tangent is constantly changing), which eliminates (a). Similar reasoning leads to the rest of the answers, which are: $(l) - (c)$, $(2) - (b)$, $(3) - (e)$, $(4) - (a)$, $(5) - (d)$.

Exercise 8.1 *Find the critical points of the function $f(x) = 3x^4 - 8x^3 + 6x^2 - 1$. Are they local maximum or minimum points?*

Solution: We begin by finding the critical points:

$$f'(x) = 12x^3 - 24x^2 + 12x = 12x(x^2 - 2x + 1) = 12x(x-1)^2;$$

the critical points are thus 0 and 1. Since $(x - 1)^2$ is always nonnegative, the only sign change is from negative to positive at 0. Thus 0 is a local minimum point, and f is increasing at 1 (see Fig. 8.3). $\square$

8.1.3 The Second Derivative and Concavity

The sign of the second derivative indicates which way the graph of a function is bending.

Example 50 *Find the intervals on which $f(x) = 3x^3 - 8x + 12$ is concave upward and on which it is concave downward. Make a rough sketch of the graph.*

Solution: Differentiating f, we get $f'(x) = 9x^2 - 8$, $f''(x) = 18x$. Thus f is concave upward when $18x > 0$ (that is, when $x > 0$) and concave downward

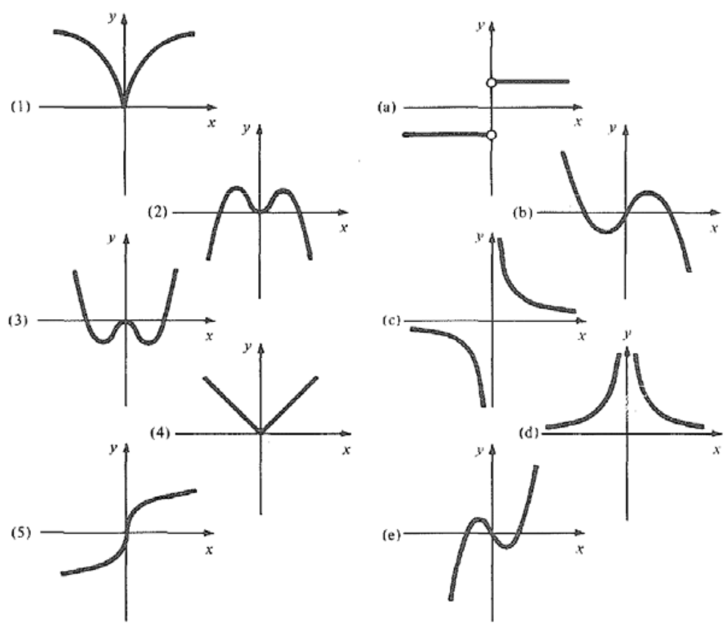


Fig. 8.2. Matching functions and their derivatives.

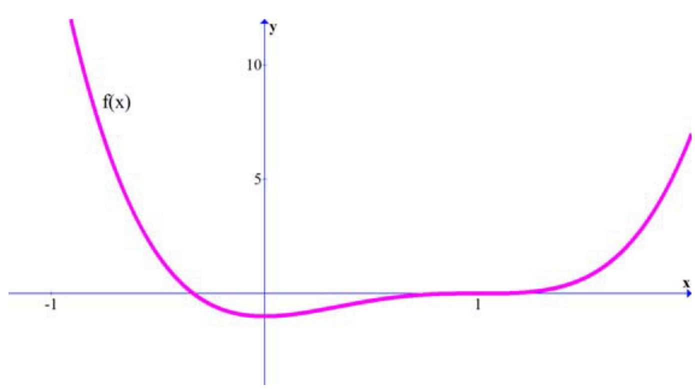


Fig. 8.3. The graph of $f(x) = 3x^4 - 8x^3 + 6x^2 - 1$.

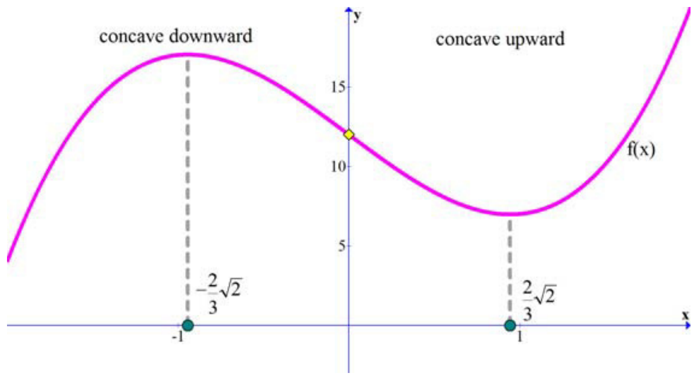


Fig. 8.4. The critical points and concavity of $f(x) = 3x^3 - 8x + 12$.

when $x < 0$. The critical points occur when $f'(x) = 0$, i.e., at $x = \pm\sqrt{8/9} = \pm \frac{2}{3}\sqrt{2}$. Since $f''(-\frac{2}{3}\sqrt{2}) < 0$, $-\frac{2}{3}\sqrt{2}$ is a local maximum, and since $f"(\frac{2}{3}\sqrt{2}) > 0$, $\frac{2}{3}\sqrt{2}$ is a local minimum. Additionaly $x = 0$ is an inflection point for f, as $f''(0) = 0$. This information is sketched in Fig. 8.4 $\square$

Example 51 *Find the inflection points of the function* $f(x) = 24x^4 - 32x^3 + 9x^2 + 1$.

Solution: We have $f'(x) = 96x^3 - 96x^2 + 18x$, so $f''(x) = 288x^2 - 192x + 18.$, Solution is: $\frac{1}{12}\sqrt{7} + \frac{1}{3}, \frac{1}{3} - \frac{1}{12}\sqrt{7}$. To find inflection points, we begin by solving $f''(x) = 0$; the quadratic formula gives $x = \left(\frac{1}{3} \pm \frac{1}{12}\sqrt{7}\right)$. Using our knowledge of parabolas, we can conclude that f'' changes from positive to negative at $\left(\frac{1}{3} - \frac{1}{12}\sqrt{7}\right)$ and from negative to positive at $\left(\frac{1}{3} + \frac{1}{12}\sqrt{7}\right)$; thus both are inflection points.

8.2 Exercises.

Exercise 8.2 *Suppose that f is continuous on $[0, 3]$, that f has no roots on the interval, and that $f(0) = 1$. Prove that $f(x) > 0$ for all x in $[0, 3]$.*

Exercise 8.3 *Where is $f(x) = 9x^2 - 3x/\sqrt{x^4 - 2x^2 - 8}$ continuous?*

Exercise 8.4 *Show that the equation $-s^5 + s^2 = 2s - 6$ has a real solution.*

Exercise 8.5 *Prove that $f(x) = x^8 + 3x^4 - 1$ has at least two distinct (real) zeros.*

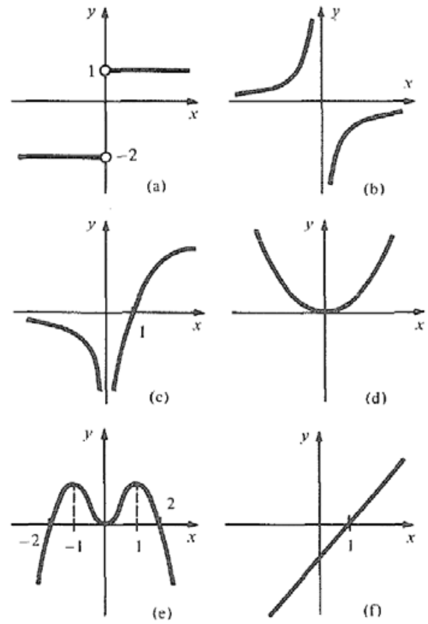


Fig. 8.5. Sketch functions that have these derivatives..

Exercise 8.6 *Find all points in which $f(x) = x^2 - 3x + 2$ is increasing, and at which it changes sign.*

Exercise 8.7 *Find a quadratic polynomial which is zero at $x = 1$, is decreasing if $x < 2$, and is increasing if $x > 2$.*

Exercise 8.8 *Sketch functions whose derivatives are shown in Fig. 8.5.*

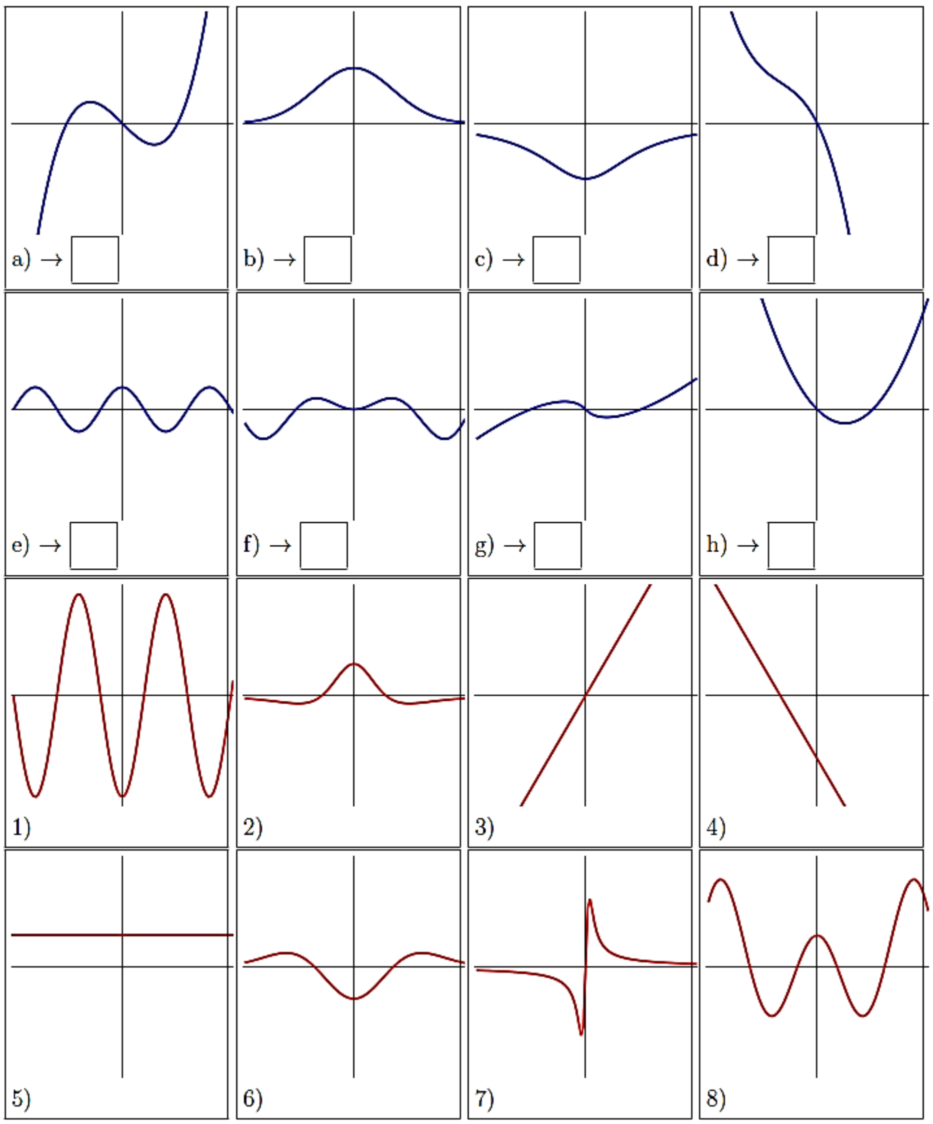
Exercise 8.9 *Find the inflection points for the following functions:*

a) $f(x) = x^3 - x$,

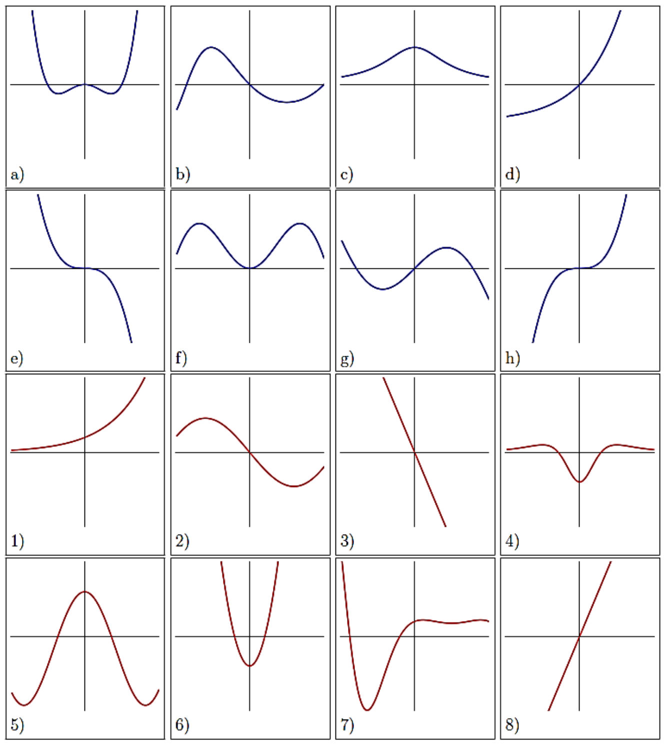
b) $f(x) = x^7$,

c) $f(x) = (x - 1)^4$.

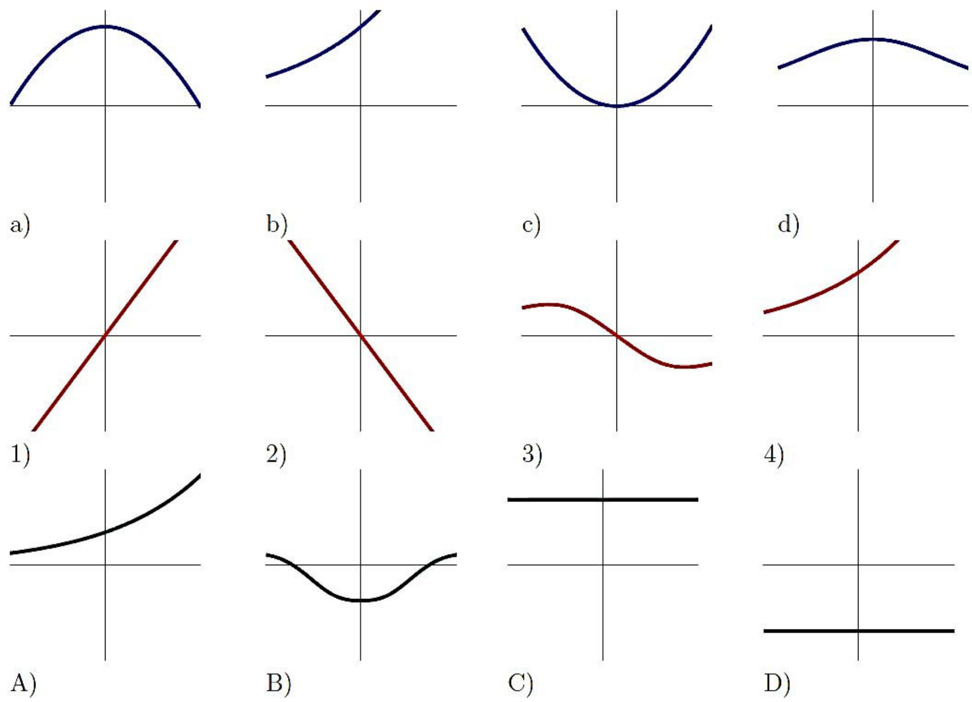
Exercise 8.10 *Match the graphs of the functions $f(x)$ in $a) - h)$ with $f''(x)$ in$1) - 8)$.*



Exercise 8.11 *Match the following functions* $a) - h)$ *with their second derivatives* $1) - 8)$*:*



Exercise 8.12 *Match the functions* (*a*-*d*) *(top row) with their derivatives* (1-4) *(middle row) and second derivatives* (*A*-*D*) *(last row).*



9

Implicit differentiation

10

Sketching graphs (Exercises)

Using calculus to determine the principal features of a graph often produces better results than simple plotting.

Graphing procedure:
To sketch the graph of a function f :

1. Note any symmetries of f. Is $f(x) = f(-x)$, or $f(x) = -f(-x)$, or neither? In the first case, f is called even; in the second case, f is called odd. If f is even that is, $f(x) = f(-x)$ we may plot the graph for $x \geq 0$ and then reflect the result across they axis to obtain the graph for $x \leq 0$. If f is odd, that is, $f(x) = -f(-x)$ then, having plotted f for $x \geq 0$, we may reflect the graph in the y axis and then in the x axis to obtain the graph for $x \leq 0$.

2. Locate any points where f is not defined and determine the behavior of f near these points. Also determine, if you can, the behavior of $f(x)$ for x very large positive and negative.

3. Locate the local maxima and minima of f, and determine the intervals on which f is increasing and decreasing.

4. Locate the inflection points of f, and determine the intervals on which f is concave upward and downward.

5. Plot a few other key points, such as x and y intercepts, and draw a small piece of the tangent line to the graph at each of the points you have plotted. (To do this, you must evaluate $f'(x)$ at each point.)

6. Fill in the graph consistent with the information gathered in steps 1 through 5.

10.1 Practice Problems

Example 52 *Sketch the graph of* $f(x) = \dfrac{x}{1+x^2}$.

Solution: We carry out the six-step procedure:

1. $f(-x) = -x/\left(1+(-x)^2\right) = -x/\left(1+(x)^2\right) = -f(x)$; f is odd, so its graph must by symmetric when reflected in the x and y axes.

2. Since the denominator $1+x^2$ is never zero, the function is defined everywhere; there are no vertical asymptotes. For $x \neq 0$, we have
$$f(x) = \frac{x}{1+x^2} = \frac{1}{x+\dfrac{1}{x}}$$
Since l/x becomes small as x becomes large, $f(x)$ looks like $1/(x+0) = l/x$ for x large. Thus $y=0$ is a horizontal asymptote; the graph is below $y=0$ for x large and negative and above $y=0$ for x large and positive.

3. $$f'(x) = \frac{1}{x^2+1} - 2\frac{x^2}{\left(x^2+1\right)^2} = \frac{\left(1-x^2\right)}{\left(x^2+1\right)^2}$$
which vanishes when $x = \pm 1$. To check the sign of $f'(x)$ on $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$, we evaluate it at conveniently chosen points: $f'(-2) = -3/25$, $\quad f'(0) = 1$, $\quad f'(2) = -3/25$. Thus f is decreasing on $(-\infty,-1)$ and on $(1,\infty)$ and f is increasing on $(-1,1)$. Hence -1 is a local minimum and 1 is a local maximum by the first derivative test.

4. $$f''(x) = 8\frac{x^3}{\left(x^2+1\right)^3} - 6\frac{x}{\left(x^2+1\right)^2} = 2\frac{x}{\left(x^2+1\right)^3}\left(x^2-3\right)$$
This is zero when $x = 0, \sqrt{3}$, and $-\sqrt{3}$. Since the denominator of f'' is positive, we can determine the sign by evaluating the numerator. Evaluating at -2, -1, 1, and 2, we get -4, 4, -4, and 4, so f is concave downward on $\left(-\infty,-\sqrt{3}\right)$ and $(0,\sqrt{3})$ and concave upward on $(-\sqrt{3},0)$and $(\sqrt{3},\infty)$; $-\sqrt{3}$, 0, and $\sqrt{3}$ are points of inflection.

5. The only solution of $f(x) = 0$ is $x = 0$.
$$\begin{aligned} f(0) &= 0 & f'(0) &= 1 \\ f(1) &= \tfrac{1}{2} & f'(1) &= 0 \\ f(\sqrt{3}) &= \tfrac{1}{4}\sqrt{3} & f'(\sqrt{3}) &= -\tfrac{1}{8} \end{aligned}$$
The information obtained in steps 1 through 5 is placed tentatively on the graph in Fig. 10.1. As we said in step 1, we need do this only for $x \geq 0$.

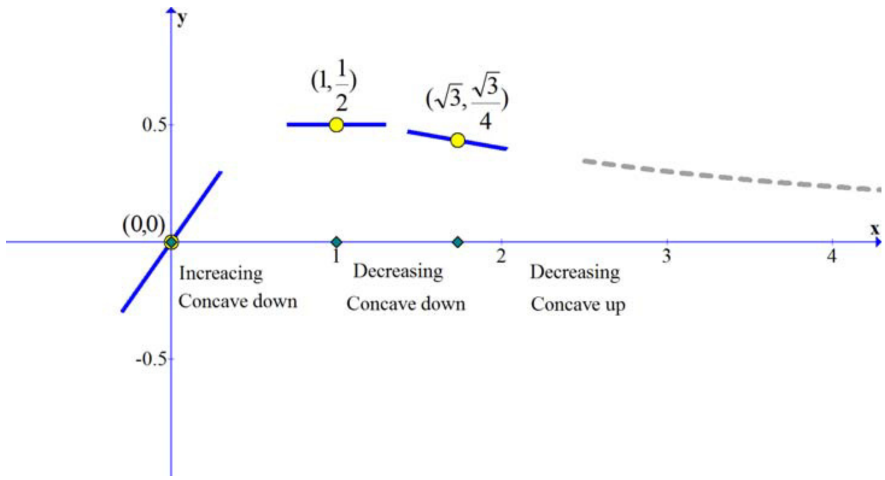


Fig. 10.1. The graph of $f(x) = \dfrac{x}{1+x^2}$ after steps 1 to 5.

6. We draw the final graph, remembering to obtain the left-hand side by reflecting the right-hand side in both axes. The result is shown in Fig. 10.2.

□

Example 53 *Sketch the graph of*

$$f(x) = (x+1)^{\frac{2}{3}}\, x^2.$$

Solution: We have

$$f'(x) = \frac{2}{3}\frac{x^2}{\sqrt[3]{x+1}} + 2x\,(x+1)^{\frac{2}{3}} = \frac{2}{3}x\frac{4x+3}{\sqrt[3]{x+1}}$$

For x near -1, but $x > -1$, $f'(x)$ is large positive, while for $x < -1$, $f'(x)$ is large negative. Since f is continuous at -1, this is a local minimum and a cusp. The other critical points are $x = 0$ and $x = -3/4$. From the first derivative test (or second derivative test, if you prefer), $-\frac{3}{4}$ is a local maximum and zero is a local minimum. For $x > 0$, f is increasing since $f'(x) > 0$; for $x < -1$, f is decreasing since $f'(x) < 0$. Thus we can sketch the graph as in Fig. 10.3. (We located the inflection points at $\backsimeq -0.208$ and -1.442 by setting the second derivative equal to zero.)

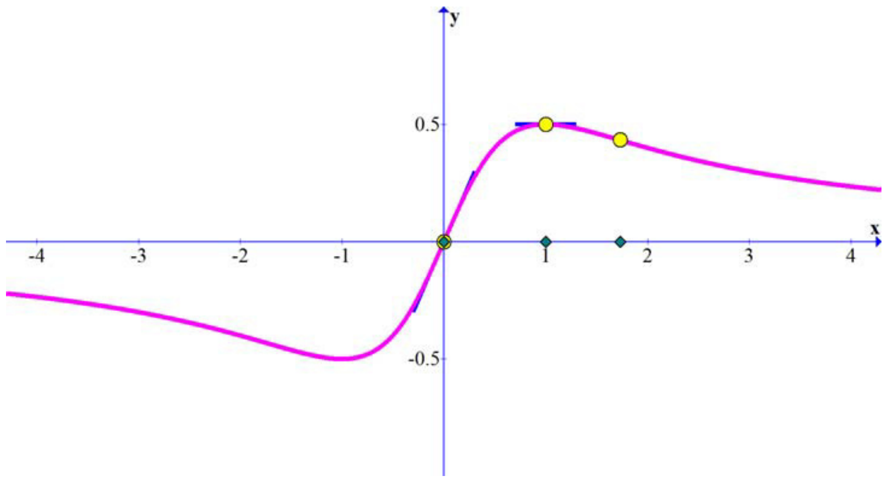


Fig. 10.2. The complete graph of $f(x) = \dfrac{x}{1+x^2}$.

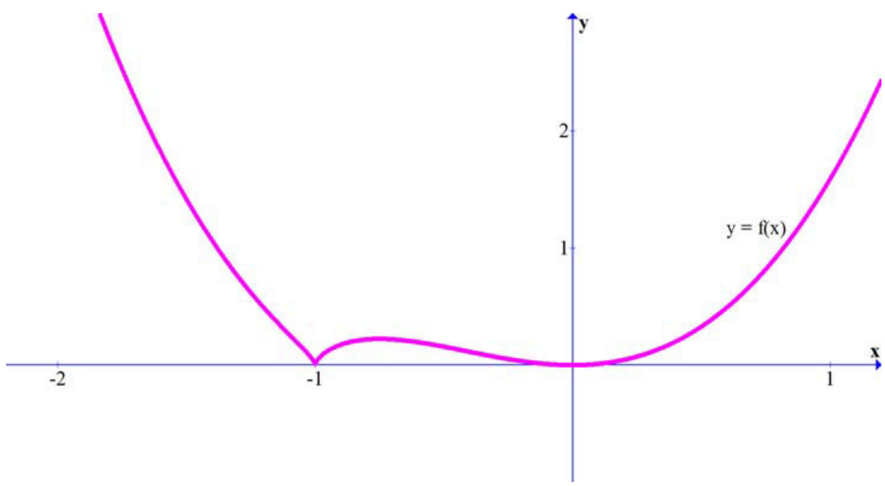


Fig. 10.3. The graph of $f(x) = (x+1)^{\frac{2}{3}}\, x^2$ has a cusp at $x = -1$.

10.2 Exercises

Exercise 10.1 *Sketch the graph of the function*

$$f(x) = \frac{1}{x^2} - \frac{1}{(x-2)^2}.$$

Indicate the asymptotes, local extrema, and points of inflection.

Exercise 10.2 *Sketch the graph of the function*

$$f(x) = 1 - \frac{3}{x} + \frac{4}{x^3}.$$

Indicate the asymptotes, local extrema, and points of inflection.

Exercise 10.3 *Sketch the graph of the function*

$$f(x) = \frac{1}{(x^2+1)^2}.$$

Indicate the asymptotes, local extrema, and points of inflect.

Exercise 10.4 *Sketch the graph* $y = x/(1-x)$ *by*

a) *the six-step procedure and*

b) *by making the transformation* $u = 1 - x$.

Exercise 10.5 *Sketch the graphs of the following functions:*

a) $f(x) = \dfrac{x^2+1}{x^2-1}$,

b) $f(x) = \dfrac{x^2-1}{x^2+1}$,

c) $f(x) = \dfrac{x^2+1}{x\,(x^2-1)}$,

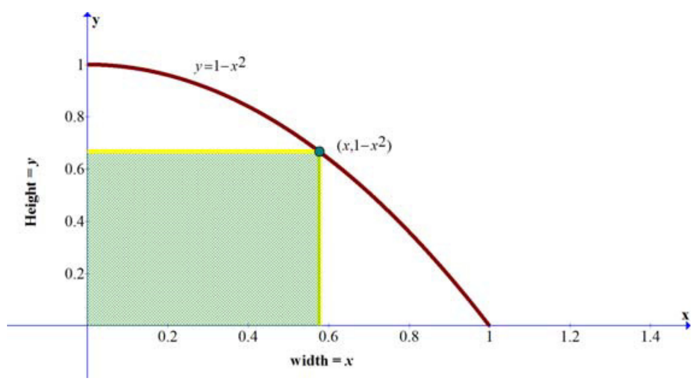
d) $f(x) = \dfrac{x^2-1}{x\,(x^2+1)}$, .

11

Optimization and linearization (Exercises)

11.1 Practice Problems

Example 54 *A rectangle is inscribed in the region in the first quadrant bounded by the coordinate axes and the parabola $y = 1 - x^2$. Find the dimensions of the rectangle that maximizes its area.*



Solution: We have

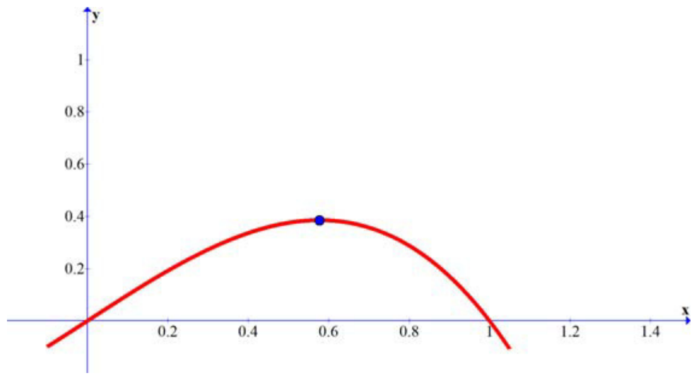
$$A = xy \qquad \text{where} \quad y = 1 - x^2$$

$$A = x\left(1 - x^2\right) \qquad \text{for} \quad 0 \le x \le 1.$$

$$A'(x) = 1 - 3x^2 = 0 \qquad \text{at} \quad x = \sqrt{\frac{1}{3}} = 0.57735$$

$$A\left(\sqrt{\frac{1}{3}}\right) = \sqrt{\frac{1}{3}}\left(1 - \frac{1}{3}\right) = \frac{2}{9}\sqrt{3} = 0.3849$$

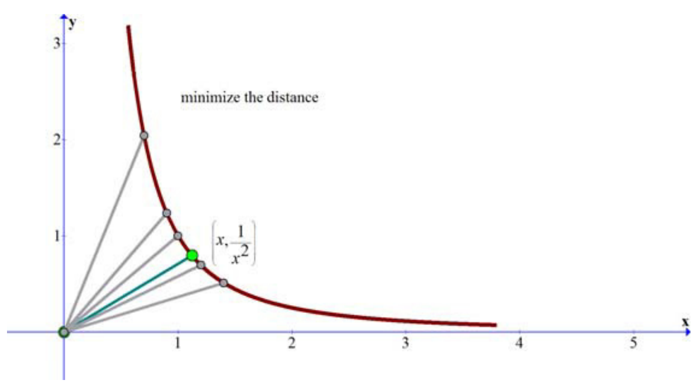
$$\text{Maximum value:} \qquad A(\sqrt{\frac{1}{3}}) = \frac{2}{9}\sqrt{3}$$



Optimal dimensions: width$=\sqrt{\frac{1}{3}}$

height$=\dfrac{2}{3}$. $\square$

Example 55 *Find the point on the graph of* $y=\dfrac{1}{x^2}$, $x>0$, *that is closest to the origin.*



Solution: Distance from the origin is given by

$$\sqrt{x^2+(1/x^2)^2}=\sqrt{x^2+1/x^4}.$$

Squared distance

$$f(x)=x^2+1/x^4=\frac{x^{6+1}}{x^4}, \quad x>0.$$

$$f'(x)=2x-\frac{4}{x^5}.$$

$$f'(x) = 0 \quad \text{at} \quad x = \sqrt[6]{2} \approx 1.1225.$$

$$f''(x) = \frac{20}{x^6} + 2 > 0.$$

Minimum value of f : $f(\sqrt[6]{2}) = \frac{3}{2}\sqrt[3]{2}$

Minimum distance: $\sqrt{f(\sqrt[6]{2})} = \sqrt{\frac{3}{2}\sqrt[3]{2}} \approx 1.3747$

Closest point on the curve: $\left(\sqrt[6]{2}, \dfrac{1}{\sqrt[6]{2}}\right)$ □

Example 56 *A juice can (in the shape of a right circular cylinder) is to have a volume of* 1 *liter (*1000 cm^3*). Find the height and radius that minimize the surface area of the can- and thus the amount of material used in its construction.*

Solution: We have

$$A = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2}$$

$$A(r) = 2\left(\pi r^2 + \frac{1000}{r}\right), \qquad 0 < r < \infty \quad \text{(domain)}.$$

$$A'(r) = 4\left(\frac{\pi r^3 - 500}{r^2}\right)$$

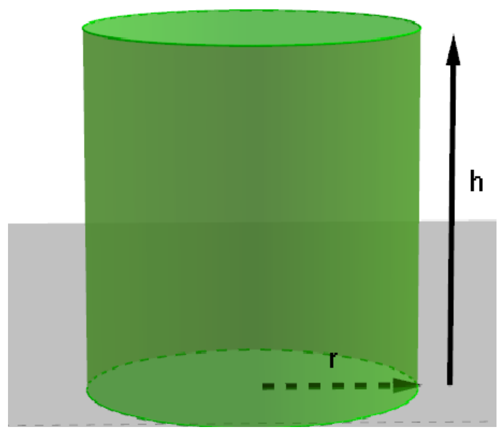
$$A'(r) = 0 \qquad \text{at} \quad r = \sqrt[3]{\frac{500}{\pi}} = 5\sqrt[3]{\frac{4}{\pi}} \quad \text{(critical number)}.$$

$$A''(r) = 4\pi + \frac{4000}{r^3} > 0.$$

Minimum area: $A(5\sqrt[3]{\frac{4}{\pi}}) = \dfrac{400}{\sqrt[3]{\frac{4}{\pi}}} + 50\pi\sqrt[3]{\frac{4}{\pi}}^2 \approx 553.58\ cm^2.$

Optimal dimensions: $r = 5\sqrt[3]{\frac{4}{\pi}} \approx 5.4193$

$$h = \frac{1000}{25\pi(4/\pi)^{2/3}} = 2r \approx 10.839.$$



□

Example 57 *An open box is to be made from a rectangular piece of cardboard that is 8 feet by3 feet by cutting out four equal squares from the corners and then folding up the flaps. What length of the side of a square will yield the box with the largest volume?*

Solution: Let x be the side of the square that is removed from each corner. The volume $V = lwh$, where l, w, and h are the length, width, and height of the box. Now $l = 8 - 2x$, $w = 3 - 2x$, and $h = x$, giving

$$V(x) = (8 - 2x)(3 - 2x)x = (4x^2 - 22x + 24)x = 4x^3 - 22x^2 + 24x.$$

The width w must be positive. Hence, $3 - 2x > 0$ or $3 > 2x$ or $\frac{3}{2} > x$. Furthermore, $x > 0$. But we also can admit the values $x = 0$ and $x = 3$, which make $V = 0$ and which, therefore, cannot yield the maximum volume. Thus, we have to maximize $V(x)$ on the interval $[0, \frac{3}{2}]$. Since

$$V'(x) = 12x^2 - 44x + 24$$

the critical numbers are the solutions of

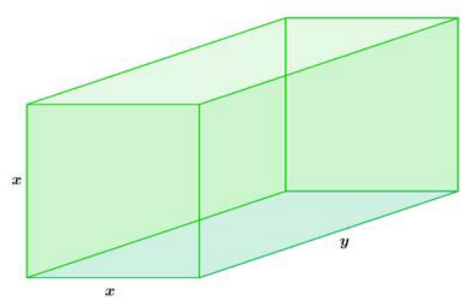
$$12x^2 - 44x + 24 = 0.$$

Solutions are: $3, \frac{2}{3}$. The only critical number in $\left(0, \frac{3}{2}\right)$ is $\frac{2}{3}$. Hence, the volume is greatest when $x = \frac{2}{3}$. □

The only critical number in (0,j) is 3. Hence, the volume is greatest when x = 3.

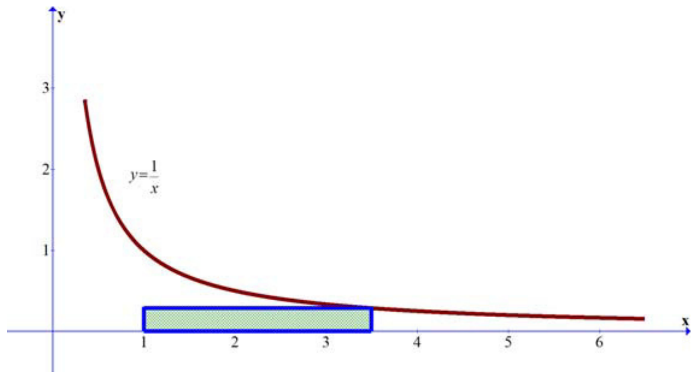
11.2 Exercises

Exercise 11.1 *An aquarium is to be built to hold* 20 *cubic feet of water. If two ends of the aquarium are square and there is no top, find the dimensions that minimize the surface area–and thus the amount of glass used in its construction.*

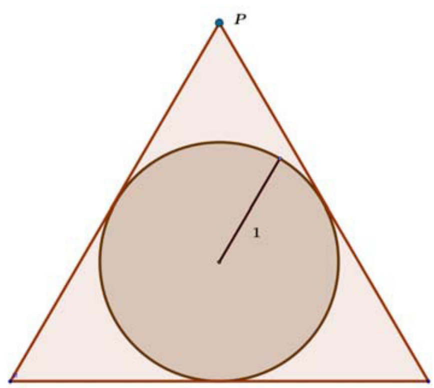


Exercise 11.2 *A rectangle is constructed in the first quadrant, as shown below, with one corner at* $(1, 0)$*, and its opposite corner on the graph of* $y = 1/x$*. Find the rectangle with largest area, or, show that there is no rectangle with*

largest area.

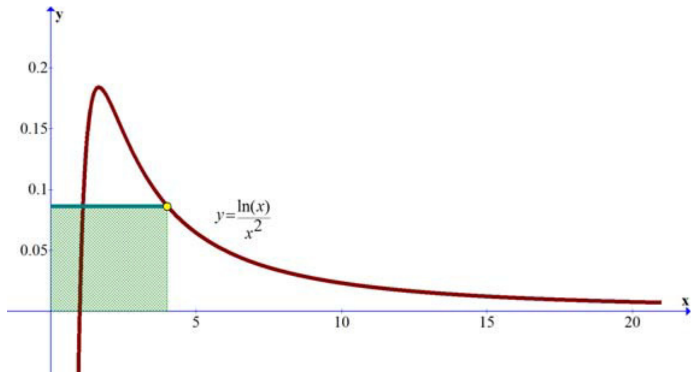


Exercise 11.3 *Suppose you can drag the blue vertex P of the triangle to create various isosceles triangles circumscribing the circle. Which isosceles triangle has minimum area? Which isosceles triangle has minimum perimeter?*



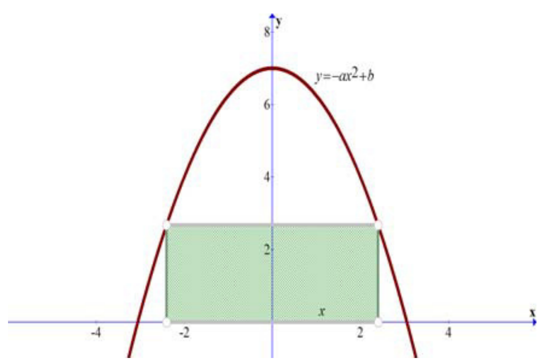
Exercise 11.4 *A rectangle is constructed in the first quadrant, as shown below, with one corner at the origin, and its opposite corner on the graph of $y = (\ln(x))/x^2$. Find the rectangle with largest area, or, show that there is no*

rectangle with largest area. Note the scale on the axes.



Exercise 11.5 *A rectangle is inscribed between the x-axis and a downward-opening parabola, as shown above. The parabola is described by the equation $y = -ax^2 + b$ where both a and b are positive.*

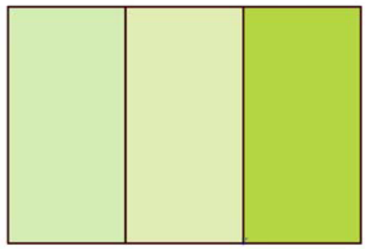
- *Let $a = 1$ and $b = 7$. What value of x maximizes the area of the rectangle?*
- *Let $a = 1$ and $b = 7$. What value of x maximizes the perimeter of the rectangle?*
- *Repeat the above two problems for a and b in general.*



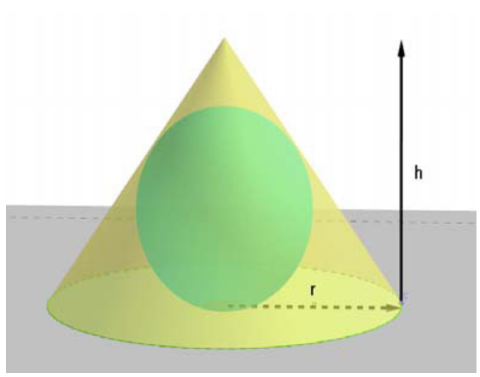
Note: x is the distance from the origin to the lower-right corner of the rectangle; x is not the length of the base of the rectangle!

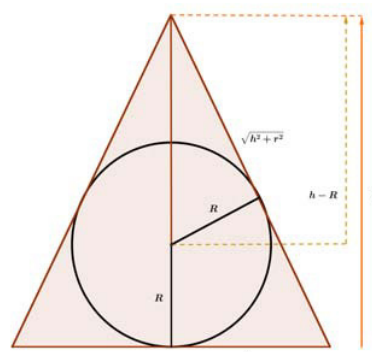
Exercise 11.6 *A farmer has exactly 1200 feet of fencing and needs to create a rectangular enclosure with three pens, as shown below. What should the*

dimensions of the enclosure be, to create the maximum area?



Exercise 11.7 *Find the dimensions of the cone with minimum volume that can contain a sphere with radius R.*





11.3 Review exercises: Chapter 11

Exercise 11.8 *Suppose a simple circuit contains two resistors with resistances R_1 and R_2 in ohms. The resistors are wired in parallel, which implies that the total resistance of the circuit is given by*

$$R_{total} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

In problems a) through c), find the minimum total resistance of the circuit given the constraint. Note that in problems like this the domain of the function is

implied by the physical reality of the situation. Specifically, $R_1 > 0$ *and* $R_2 > 0$ *is implied if not explicitly stated. A constraint is used to eliminate either* R_1 *or* R_2 *from the problem, leaving you with a function of a single variable.*

a) *The sum of the resistances of the two resistors is* 30 *ohms.*

b) *The sum of the resistances of the two resistors is* Ω *ohms,* $\Omega > 0$.

c) *The resistance of one resistor is twice the resistance of the second resistor.*

Exercise 11.9 *More generally, a parallel circuit consisting of* n *resistors has total resistance*

$$R_{total} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + ... + \frac{1}{R_n}}$$

where R_k *is the resistance of the* k*th resistor. Suppose that the sum of the resistances in three resistors is* 120 *ohms and that the resistance of one of the resistors is twice the resistance of one of the others. Calculate the minimum total resistance in the circuit.*

Exercise 11.10 *Suppose we want to build a rectangular storage container with a volume of* 12 *cubic meters. Assume that the cost of materials for the base is* \$12 *per square meter, and the cost of materials for the sides is* \$8 *per square meter. The height of the box is three times the width of the base. What's the least amount of money we can spend to build such a container?*

Exercise 11.11 *Suppose you wish to construct a right, circular, cylindrical cup (without a top) that will hold* 1000 *cubic centimeters of liquid. What dimensions should the cup have in order to minimize the total amount of material used to construct the cup? Assume that the cup is arbitrarily thin, so that it is the surface area that we wish to minimize.*

Exercise 11.12 *As in the previous problem, suppose you wish to construct a right, circular, cylindrical cup with no top that will hold* 1000 *cubic centimeters of liquid. Assume that the base needs to be built out of higher quality material than the rest of the cup, perhaps because the cup will hold highly acidic liquid. Material for the base therefore costs* \$8 *per square centimeter while material for the rest of the cup costs* \$6 *per square centimeter. Find the dimensions of the cup that minimize the cost of materials.*

Exercise 11.13 *Suppose that* 100 *meters of fencing are used to build a rectangular enclosure against a very long wall (so that no fencing needs to be used along the wall).*

a) *What is the largest area of the enclosure that can be constructed using the fence?*

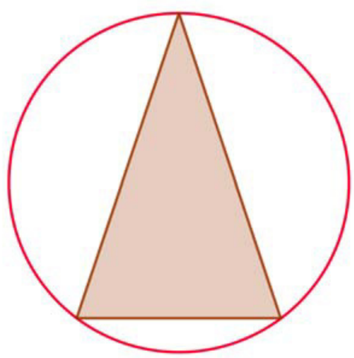
b) *What are the dimensions of the maximum-area enclosure?*

Exercise 11.14 *A farmer is building a rectangular enclosure for* 4000 *square yards of land. The enclosure is to be split lengthwise into three rectangular pens of equal area. The outer fencing for the enclosure costs* \$10 *per yard and the inner partitions cost* \$5 *per yard.*

a) *Find the dimensions of the enclosure that would minimize the total cost.*

b) *Find the dimensions of the enclosure that would minimize the total fencing used.*

Exercise 11.15 *An isosceles triangle is inscribed in a circle of radius* 2 *as shown in the diagram below. Find the dimensions of the triangle with largest area inside the circle. (*`HINT`*: Let x denote half the length of the base, and show that the height of the triangle is then $2 + \sqrt{4 - x^2}$).*

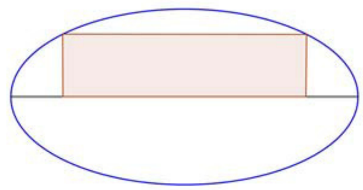


Exercise 11.16 *Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.*

Exercise 11.17 *A rectangle is inscribed in the upper half of the ellipse.*

$$\frac{x^2}{4} + y^2 = 1$$

as shown in the diagram. Find the dimensions of the rectangle with maximal area drawn in this fashion.



Exercise 11.18 *Find the maximum perimeter of a rectangle that can be inscribed in the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Exercise 11.19 *Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius* 6 *inches and height* 10 *inches.*

Exercise 11.20 *Suppose that to make nsunglasses, it costs a company $f(n)$ dollars, where $f(n) = n^3 + 2n + 2000$.*

a) *What are the fixed costs associated with production of the sunglasses, i.e., what are the costs that the company incurs for setting up to produce the sunglasses, even if they don't actually produce any?*

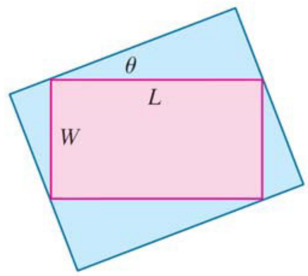
b) *Denote $A(n)$ to be the function giving the average cost per pair of sunglasses to produce n sunglasses. Find an expression for $A(n)$, and determine the interval on which it should be defined.*

c) *How many sunglasses should be made to minimize the average cost per pair of sunglasses? What is the minimum average cost?*

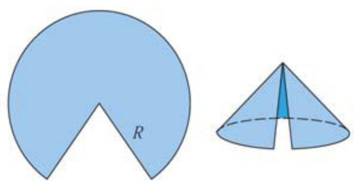
Exercise 11.21 *A piece of wire* 10 *m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?*

Exercise 11.22 *Find an equation of the line through the point* $(3, 5)$ *that cuts off the least area from the first quadrant.*

Exercise 11.23 *Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length L and width W.*



Exercise 11.24 *A cone is made from a circular sheet of radius R by cutting out a sector and gluing the cut edges of the remaining piece together. What is the maximum volume attainable for the cone?*



Exercise 11.25 *According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed* 108 *inches, where by "girth" we mean the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square end that can be sent by mail? What are the dimensions of the package of largest volume?*

Exercise 11.26 *A closed box has a fixed surface area A and a square base with side x.*

a) *Find a formula for the volume, V, of the box as a function of x. What is the domain of V ?*

b) *Graph V against x.*

c) *Find the maximum value of V.*

Exercise 11.27 *A rectangular swimming pool is to be built with an area of* 1800 *square meters. The owner wants* 5*-meter wide decks along either side and* 10*-meter wide decks at the two ends. Find the dimensions of the smallest piece of property on which the pool can be built satisfying these conditions.*

Exercise 11.28 *A square-bottomed box with a top has a fixed volume, V. What dimensions minimize the surface area?*

Exercise 11.29 *Find the global maxima and minima of the function*

$$f(x) = 3|x| - x^3$$

on the interval $[-1, 2]$. Sketch a graph.

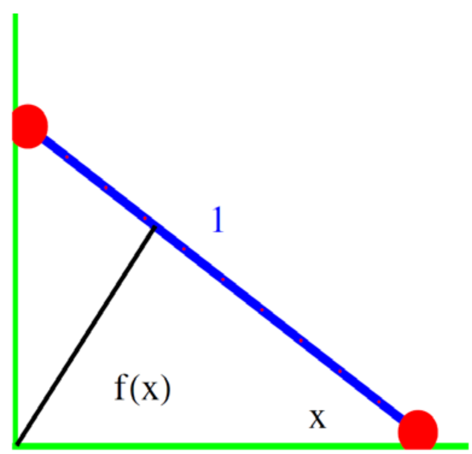
Exercise 11.30 *Find the largest area $A = 4xy$ of a rectangle with vertices (x, y), $(-x, y)$, $(-x, -y)$, $(x, -y)$ inscribed in the ellipse*

$$x^2 + 2y^2 = 1.$$

Exercise 11.31 *A ladder of length 1 is one side at a wall and on one side at the floor.*

a) *Verify that the distance from the ladder to the corner is $f(x) = sin(x)cos(x)$.*

b) *Find the angle x for which $f(x)$ is maximal.*



Exercise 11.32 *A candy manufacturer builds spherical candies. Its effectiveness is $A(r) - V(r)$, where $A(r)$ is the surface area and $V(r)$ the volume of a candy of radius r. Find the radius, where $f(r) = A(r) - V(r)$ has a global*

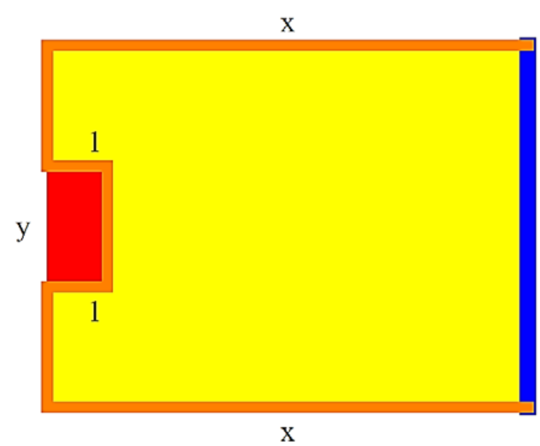
maximum for $r \geq 0$.

Exercise 11.33 *A tennis field of width x and length y contains a fenced referee area of length 2 and width 1 within the field and an already built wall. The circumference a fence satisfies $2x+y+2=100$, (an expression which still can be simplified). We want to maximize the area $xy-2$.*

a) *On which interval [a, b] does the variable x make sense? Find a function f(x) which needs to be maximized.*

b) *Find the local maximum of x and check it with the second derivative test.*

c) *What is the global maximum of f on $[a,b]$?*



CALCULUS MIDTERM 17.12.2014 ∗ GROUP 3

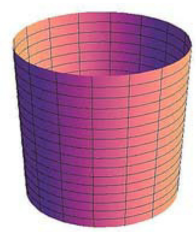
Name: ____________________

- *Please write neatly. Answers which are illegible for the grader can not be given credit.*

- *No calculators, computers, or other electronic aids are allowed.*

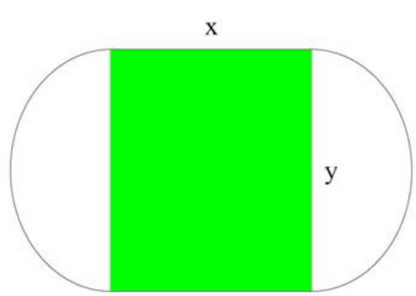
- *You have 90 minutes time to complete your work.*

Exercise 11.34 *A cup of height h and radius r has the volume $V = \pi r^2 h$. Its surface area is $\pi r^2 + \pi r h$. Among all cups with volume $V = \pi$ find the one which has minimal surface area. Find the global minimum.*

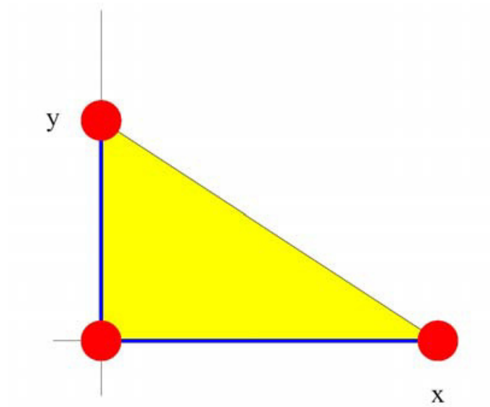


Exercise 11.35 *Find a concrete function which has three local maxima and two local minima.*

Exercise 11.36 *The University stadium has a track which encloses a rectangular field of dimensions x, y. The circumference of the track is $400 = 2\pi y + 2x$ and is fixed. We want to maximize the area xy for a play field. Which x achieves this?*



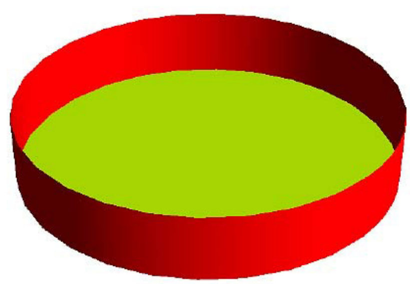
Exercise 11.37 *Which rectangular triangle* $(0,0)$, $(x,0)$, $(0,y)$ *with* $x+y=2$, $x \geq 0$, $y \geq 0$ *has maximal area* $A = xy/2$*?*



Exercise 11.38 *A candle holder of height* y *and radius* x *is made of aluminum. Its total surface area is* $2\pi xy + \pi x^2 = \pi$ *(implying* $y = 1/(2x) - x/2$*). Find* x *for which the volume*

$$f(x) = x^2 y(x)$$

is maximal.



12

Definite integrals (Exercises)

12.1 Practice Problems

12.1.1 The Definite Integral of a Continuous Function

Example 58 *Find $L_f(P)$ (lower Riemann sum) and $U_f(P)$ (lower Riemann sum) for $f(x) = x^2$, $x \in [0,1]$, and partition $P = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$*

Solution: The function $f(x) = x^2$ is continuous on the closed interval $[0,1]$. The partition P divides the interval $[0,1]$ into two subintervals $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ with the following lengths:

$$\begin{aligned}
\Delta x_1 &= \frac{1}{4} - 0 = \frac{1}{4} \\
\Delta x_2 &= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}. \\
\Delta x_3 &= 1 - \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

Since f is increasing on $[0,1]$, f attains its maximum value at the right endpoint of each subinterval and its minimum value at the left endpoint of each subinterval.

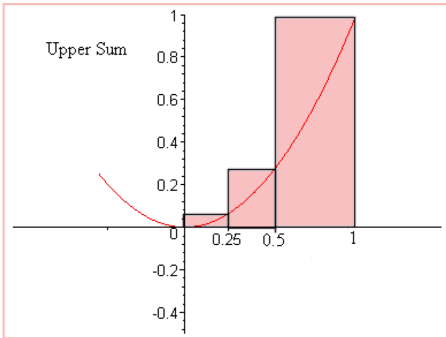
$$M_1 = f(\frac{1}{4}) = \frac{1}{16} \qquad m_1 = f(0) = 0$$

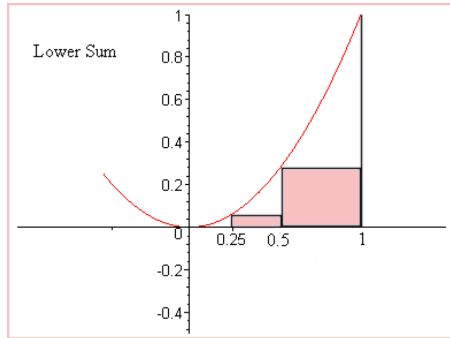
$$M_2 = f(\frac{1}{2}) = \frac{1}{4} \qquad m_2 = f(\frac{1}{4}) = \frac{1}{16}$$

$$M_3 = f(1) = 1 \qquad m_3 = f(\frac{1}{2}) = \frac{1}{4}$$

$$U_f(F) = M_1\Delta x_1 + M_2\Delta x_2 + M_3\Delta x_3 = \frac{1}{16}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4} + 1\cdot\frac{1}{2} = \frac{37}{64} = 0.578\,13,$$

$$L_f(F) = m_1\Delta x_1 + m_2\Delta x_2 + m_3\Delta x_3 = 0\cdot\frac{1}{4} + \frac{1}{16}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2} = \frac{9}{64} = 0.140\,63.$$





□

Example 59 *Use upper and lower sums to show that*

$$0.6 < \int_0^1 \frac{1}{1+x^2} dx < 1$$

Solution: The function $f(x) = \frac{1}{1+x^2}$ is continuous on the closed interval $[0, 1]$. Let $P = \{0, \frac{1}{2}, 1\}$. The partition P divides the interval $[0, 1]$ into two subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ with the following lengths:

$$\begin{aligned} \Delta x_1 &= \frac{1}{2} - 0 = \frac{1}{2} \\ \Delta x_2 &= 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Since f is decreasing on $[0, 1]$, f attains its maximum value at the left endpoint of each subinterval and its minimum value at the right endpoint of each subinterval.

$$M_1 = f(0) = 1 \qquad m_1 = f(\frac{1}{2}) = \frac{4}{5}$$

$$M_2 = f(\frac{1}{2}) = \frac{4}{5} \qquad m_2 = f(1) = \frac{1}{2}$$

Thus

$$\begin{aligned} U_f(F) &= M_1\Delta x_1 + M_2\Delta x_2 \\ &= 1 \cdot \frac{1}{2} + \frac{4}{5} \cdot \frac{1}{2} = \frac{9}{10} = 0.9. \end{aligned}$$

$$\begin{aligned} L_f(F) &= m_1\Delta x_1 + m_2\Delta x_2 \\ &= \frac{4}{5} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{13}{20} = 0.65. \end{aligned}$$

Thus,

$$0.6 < 0.65 = L_f(F) \leq \int_0^1 \frac{1}{1+x^2} dx \leq U_f(F) = 0.9 < 1$$

□

Example 60 *Given that*

$$\int_0^1 f(x)dx = 4, \qquad \int_0^3 f(x)dx = 2 \qquad \textit{and} \qquad \int_3^6 f(x)dx = 1$$

find each of the following:

a) $\int_0^6 f(x)dx$,

b) $\int_1^3 f(x)dx$,

c) $\int_3^0 f(x)dx$,

d) $\int_3^3 f(x)dx$.

Solution:

a) $\int_0^6 f(x)dx = \int_0^3 f(x)dx + \int_3^6 f(x)dx = 2 + 1 = 3$

b) $\int_1^3 f(x)dx = \int_1^0 f(x)dx + \int_0^3 f(x)dx = -\int_0^1 f(x)dx + \int_0^3 f(x)dx = -4 + 2$ $= -2$

c) $\int_3^0 f(x)dx = -\int_0^3 f(x)dx = -2$

d) $\int_3^3 f(x)dx = 0$

□

12.1.2 Indefinite integral

Exercise 12.1 *Find the indefinite integral*

$$\int \left(\frac{x^3-2}{x^2}\right) dx.$$

Solution:

$$\int \left(\frac{x^3-2}{x^2}\right) dx = \int \left(x - 2x^{-2}\right) dx = \frac{1}{2}x^2 - \frac{2}{(-1)}x^{-1} + C = \frac{1}{2}x^2 + \frac{2}{x} + C$$

□

Exercise 12.2 *Find $f(x)$ from the given information:*

$$f''(x) = x^2 - x, \quad f'(1) = 2, \quad f(1) = 2.$$

Solution:

$$f'(x) = \int \left(x^2 - x\right) dx = \frac{1}{6}x^2 (2x - 3) + C.$$

To evaluate the constant K, we use the fact that $f'(1) = 2$. Since $f'(1) = 2$ and

$$f'(1) = \frac{1}{6}(1) \cdot 1^{2\cdot} (2 \cdot 1 - 3) + C = -\frac{1}{6} + C,$$

$$C = \frac{13}{6}.$$

Therefore

$$f'(x) = \frac{1}{6}x^2 (2x - 3) + \frac{13}{6} = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{13}{6}$$

and

$$f(x) = \int \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{13}{6}\right) dx = \frac{1}{12}x^4 - \frac{1}{6}x^3 + \frac{13}{6}x + K.$$

To evaluate the constant K, we use the fact that $f(1) = 3$. Since $f(1) = 3$ and

$$f(1) = \frac{1}{12}(1)^4 - \frac{1}{6}(1)^3 + \frac{13}{6}(1) + K = K + \frac{25}{12},$$

$$K = \frac{11}{12}.$$

Therefore,

$$f(x) = \frac{1}{12}x^4 - \frac{1}{6}x^3 + \frac{13}{6}x + \frac{11}{12}.$$

□

Example 61 *Compute the following indefinite integrals:*

a) $\int \left(5x^2 - 4x + 1\right) dx$

b) $\int \left(8x^3 - \sqrt{3x}\right) dx$

c) $\int \dfrac{t^3 + 1}{t^2} dt$

d) $\int (y - 3)\sqrt{y}dy$

Solution:

a)

$$\begin{aligned}\int \left(5x^2 - 4x + 1\right) dx &= \int 5x^2 dx + \int (-4x)\, dx + \int 1 dx \\ &= 5\int x^2 dx - 4\int x dx + \int dx \\ &= \frac{5}{3}x^3 - 2x^2 + x + C\end{aligned}$$

b)

$$\int \left(8x^3 - \sqrt{3x}\right) dx = \int \left(8x^3 - \sqrt{3}x^{\frac{1}{2}}\right) dx = 2x^4 - \frac{2}{3}\sqrt{3}x^{\frac{3}{2}} + C$$

c)

$$\int \frac{t^3+1}{t^2} dt = \int \left(t + t^{-2}\right) dt = \frac{1}{2t}\left(t^3 - 2\right) + C$$

d)

$$\begin{aligned}\int (y-3)\sqrt{y} dy &= \int \left(y^{\frac{3}{2}} - 3y^{\frac{1}{2}}\right) dy \\ &= \frac{2}{5}y^{\frac{3}{2}}(y-5) + C\end{aligned}$$

□

Example 62 *A particle moves along the x-axis with velocity $v(t) = At^2 + 1$. Determine A given that $x(0) = x(1)$. Find the total distance traveled by the particle during the first second.*

Solution:

$$x(t) = \int \left(At^2 + 1\right) dt = \frac{1}{3}At^3 + t + C.$$

To determine A, use the fact that $x(0) = x(1)$.

$$x(0) = \frac{1}{3}A(0)^3 + (0) + C = C,$$

$$x(0) = \frac{1}{3}A(1)^3 + (1) + C = \frac{A}{3} + 1 + C.$$

So

$$C = \frac{A}{3} + 1 + C,$$

and it follows, that

$$A = -3,$$

and

$$v(t) = v(t) = -3t^2 + 1.$$

To find the total distance traveled by the particle during the first second, evaluate $\int_0^1 |v(t)|\, dt$:

$$\begin{aligned}\int_0^1 |v(t)|\, dt &= \int_0^{1/\sqrt{3}} \left(-3t^2 + 1\right) dt + \int_{1/\sqrt{3}}^1 \left(3t^2 - 1\right) dt \\ &= \frac{2}{9}\sqrt{3} + \frac{2}{9}\sqrt{3} = \frac{4}{9}\sqrt{3} \approx: 0.7698.\end{aligned}$$

The total distance traveled by the particle during the first second is $\frac{4}{9}\sqrt{3}$.

12.2 Exercises

Exercise 12.3 *Find $L_f(P)$ and $U_f(P)$ for $f(x) = x^2, \quad x \in [-1, =], \quad P = \{-1, -\frac{3}{4}, -\frac{1}{4}, 0\}$*

`Answer:` $U_f(F) = \frac{35}{64,} \quad L_f(F) = \frac{11}{64}.$

Exercise 12.4 *Let $f(x) = x^2$, $x \in [0, 4]$, $x_1^* = 1$, $x_2^* = 2$, $x_3^* = 3$ and $x_4^* = 4$. Draw a figure showing the Riemann Sum $S^*(P)$. Compute the value of $S^*(P)$.*

`Answer:` $S^*(P) = 30.$

Exercise 12.5 *Find the value of*

a) $\sum_{k=1}^n f(x_k^*)\Delta x_k$,

b) $\max \Delta x_k$

when

1. $f(x) = x + 1$; $a = 0$, $b = 4$;$n = 3$; $\Delta x_1 = 1$, $\Delta x_2 = 1$, $\Delta x_3 = 2$; $x_1^* = \frac{1}{3}$, $x_2^* = \frac{2}{3}$, $x_3^* = 3$;
2. $f(x) = \cos x$; $a = 0$, $b = 2\pi$; $n = 4$; $\Delta x_1 = \pi/2$, $\Delta x_2 = 3\pi/4$, $\Delta x_3 = \pi/2$, $\Delta x_4 = \pi/4$; $x_1^* = \pi/4$, $x_2^* = \pi$, $x_3^* = 3\pi/3$, $x_4^* = 7\pi/4$;
3. $f(x) = x^3$; $a = -3$, $b = 3$; $n = 4$;$\Delta x_1 = 2$, $\Delta x_2 = 1$, $\Delta x_3 = 1$, $\Delta x_4 = 2$; $x_1^* = -2$, $x_2^* = 0$, $x_3^* = 0$, $x_4^* = 2$;

Exercise 12.6 *Use the given values of a and b to express the following limits as integrals. (Do not evaluate the integrals.)*

a) $\lim_{\Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^2 \Delta x_k, \qquad a=-1,\ b=2, \quad x_{k-1} \le x_k^* \le x_k,$

b) $\lim_{\Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^3 \Delta x_k, \qquad a=1,\ b=2, \quad x_{k-1} \le x_k^* \le x_k,$

c) $\lim_{\Delta x_k \to 0} \sum_{k=1}^{n} 4x_k^*(1-3x_k^*)^3 \Delta x_k, \qquad a=-3,\ b=3, \quad x_{k-1} \le x_k^* \le x_k,$

d) $\lim_{\Delta x_k \to 0} \sum_{k=1}^{n} \sin{(x_k^*)}^2 \, \Delta x_k, \qquad a=0,\ b=\pi, \quad x_{k-1} \le x_k^* \le x_k,$

Exercise 12.7 *Express the integrals as limits of Riemann sums. (Do not evaluate the integrals.)*

a) $\int_1^2 x^2 dx,$

b) $\int_1^2 \sqrt{x} dx,$

c) $\int_0^1 \frac{x}{2+x} dx,$

d) $\int_0^{\pi/2} \cos(3x) dx,$

e) $\int_{-1}^{1} \frac{x^2+1}{x^3-2} dx.$

Exercise 12.8 *Sketch the region whose signed area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry, where needed*

a) $\int_1^3 x dx,$

b) $\int_{-3}^{-1} x dx,$

c) $\int_{-1}^{4} x dx,$

d) $\int_0^3 \left(1 - \frac{1}{2}x\right) dx,$

e) $\int_{-1}^{2} |2x-3| \, dx,$

f) $\int_{-1}^{1} \sqrt{1-x^2} dx,$

g) $\int_0^2 \sqrt{4-x^2} dx$

Exercise 12.9 *In each part, evaluate the integral, given that*

$$f(x) = \begin{cases} |x-2|, & x \ge 0 \\ x+2, & x < 0. \end{cases}$$

a) $\int_{-2}^{0} f(x)dx$,

b) $\int_{0}^{6} f(x)dx$,

c) $\int_{0}^{6} f(x)dx$,

d) $\int_{-2}^{2} f(x)dx$,

e) $\int_{-2}^{4} f(x)dx$,

Exercise 12.10 *Define a function f on $[0,1]$ by*

$$f(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & x = 0. \end{cases}$$

Use Definition **??** *to show that*

$$\int_0^1 f(x)dx = 1.$$

Exercise 12.11 *Evaluate the limit by expressing it as a definite integral over the interval $[a,b]$ and applying appropriate formulas from geometry.*

a) $\lim_{\Delta x_k \to 0} \sum_{k=1}^{n} (3x_k^* + 1)\Delta x_k$, $a = 0$, $b = 2$.

b) $\lim_{\Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{4 - x_k^{*2}}\Delta x_k$, $a = -2$, $b = 2$.

Exercise 12.12 *In each part, use Theorem***??** *to determine whether the function f is integrable on the interval $[-1,1]$.*

a) $f(x) = \cos(x)$,

b) $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

c) $f(x) = \begin{cases} \dfrac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

d) $f(x) = \begin{cases} \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$.

Exercise 12.13 *Find the indefinite integral*

$$\int \left(2\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$$

`Answer:` $\frac{2}{3}\sqrt{x}\,(2x+3)\ +C.$

Exercise 12.14 *An object moves along a coordinate line with velocity* $v(t) = 6t^2 - 6$ *units per second. Its initial position is* 2 *units to the left of the origin.*

a) *Find the position of the object* 3 *seconds later,*

b) *Find the total distance traveled by the object during those* 3 *seconds.*

`Answer:`

a) The position of the object 3 seconds later is 34 units to the right of the origin.

b) The total distance traveled by the object during those 3 seconds is 44 units.

13

The fundamental theorem of calculus (Exercises)

13.1 Practice Problems

13.1.1 The Function $F(x)=\int_a^x f(t)dt$

Example 63 *For* $F(x) = \int_0^x \sqrt{t^2+4}dt$ *find the following:*

a) $F'(-1)$,

b) $F'(0)$,

c) $F'(\frac{1}{2})$,

d) $F''(x)$.

Solution:

a) $F'(-1) = \sqrt{5}$,

b) $F'(0) = 2$,

c) $F'(\frac{1}{2}) = \frac{1}{2}\sqrt{17}$,

d) $F''(x) = \dfrac{x}{\sqrt{x^2+4}}$.

Example 64 *For* $F(x) = \int_x^0 (t+2)^3 dt = -\int_0^x (t+2)^3 dt$ *find the following:*

a) $F'(-1)$,

b) $F'(0)$,

c) $F'(\frac{1}{2})$,

d) $F''(x)$

Solution: $F(x) = \int_x^0 (t+2)^3 dt = -\int_0^x (t+2)^3 dt \ , \qquad F'(x) = -(x+2)^3$

a) $F'(-1) = -(-1+2)^3 = -1$,

b) $F'(0) = -(0+2)^3 = -8$,

c) $F'(\frac{1}{2}) = -(\frac{1}{2}+2)^3 = -\left(\frac{5}{2}\right)^3 = -\frac{125}{8}$,

d) $F''(x) = -3(x+2)^2$.

□

Example 65 *Let $F(x) = \int_0^{x^3} \cos t dt$. Find the derivative of F.*

Solution: **Let** $u(x) = x^3$ and use the Chain Rule to find the Derivative of F.

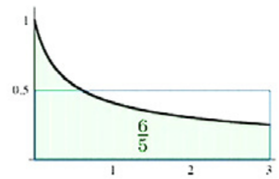
$$F'(x) = \frac{d}{dx}\left(\int_0^u \cos t dt\right)\frac{d}{dx}\left(x^3\right) = (\cos u)\left(3x^2\right) = 3x^2 \cos x^3$$

□

Example 66 *Find area under the graph of*

$$f(x) = \frac{1}{\sqrt{5x+1}}$$

between $x = 0$ and $x = 3$.



Solution: First we can find the indefinite integral

$$\begin{aligned}\int (5x+1)^{-1/2} dx &= \frac{1}{5}2\int 5\frac{1}{2}(5x+1)^{-1/2} \\ &= \frac{2}{5}(5x+1)^{1/2} + C \\ &= \frac{2}{5}\sqrt{5x+1} + C\end{aligned}$$

Next we can easily compute the definite integral

$$\begin{aligned}\int_0^3 (5x+1)^{-1/2} dx &= \frac{2}{5}\sqrt{5x+1}|_0^3 \\ &= \frac{2}{5}\left(\sqrt{5x+1}|_0^3\right) \\ &= \frac{2}{5}\left(\sqrt{16} - \sqrt{1}\right) = \frac{6}{5}\end{aligned}$$

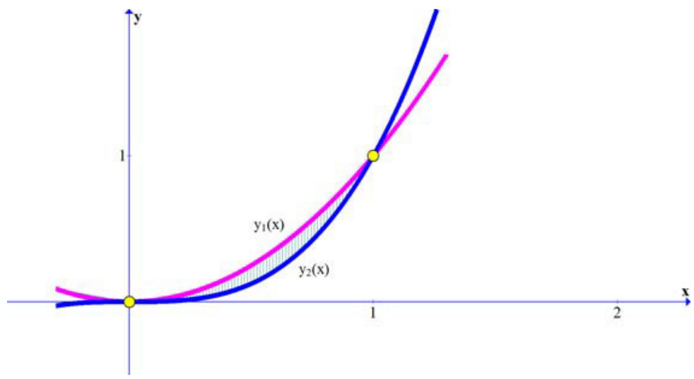
□

13.1.2 Area of a Region between 2 Curves

If $f(x) \geq g(x)$ on the interval $[a, b]$, we say that the small rectangle of width Δx and height $f(x) - g(x)$ is a **representative rectangle**.

Example 67 *Find the area of the region in the first quadrant bounded by the graphs of the curves $y_1(x) = x^2$ and $y_2(x) = x^3$.*

Solution: By setting the equations equal to each other, you see that the curves intersect at $(0, 0)$ and $(1, 1)$. Sketch the region under consideration, noting that the curve $y_1(x) = x^2$ is above the other curve. Draw a representative rectangle on the region between the curves. The base of the rectangle has length Δx, and the height is $y_1(x) - y_2(x) = x^2 - x^3$.



Loosely speaking, the area between the curves is obtained by adding up the representative rectangles. That is, the area is the following definite integral

$$Area = \int_0^1 (y_1(x) - y_2(x))\, dx = \int_0^1 \left(x^2 - x^3\right) dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)|_0^1 = \frac{1}{12}.$$

□

Sometimes you encounter area problems of more complicated regions. For instance, the 2 curves might intersect at more than 2 points. In this case, you must find all the points of intersection and determine which curve is above the other on each interval determined by these points.

Example 68 *Find the area of the region between the graphs $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.*

Solution: Set the equations equal to each other to find the points of intersection.

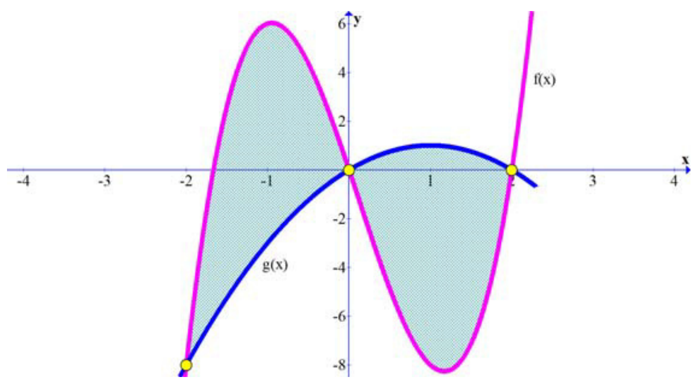
$$3x^3 - x^2 - 10x = -x^2 + 2x$$

$$3x^3 - 12x = 0$$

$$3x\,(x-2)\,(x+2) = 0$$

$$x = 0, \quad -2, \quad 2.$$

The curves intersect at 3 points: $(0,0), (2,0)$, and $(-2,-8)$.



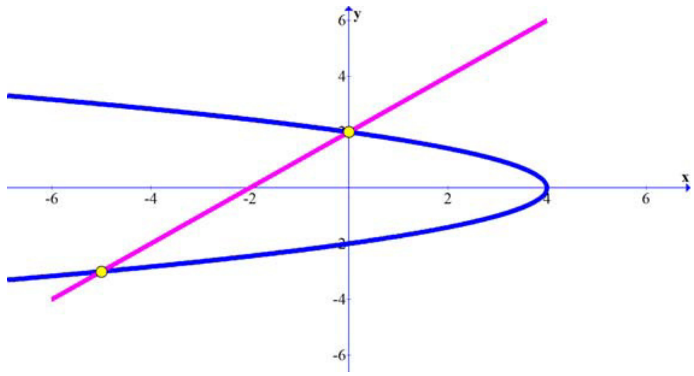
On the interval $-2 \leq x \leq 0$, the graph of f is above that of g , whereas on the interval $0 \leq x \leq 2$, the graph of g is above that of f . Hence, the area is given by the 2 integrals shown below

$$\begin{aligned} Area &= \int_{-2}^{0} (f(x) - g(x))\,dx + \int_{0}^{2} (g(x) - f(x))\,dx \\ &= 12 + 12 = 24. \end{aligned}$$

□

Example 69 *Consider the area bounded by* $x = 4 - y^2$ *and* $x = y - 2$ *. Calculate this area by integrating with respect to* x*, and then with respect to* y*. Which method is simpler?*

Solution: Setting the equations equal to each other, you see that they intersect at the points $(0, 2)$ and $(-5, -3)$.



You will need 2 integrals if you integrate with respect to x :

$$\int_{-5}^{0} \left((x+2) + \sqrt{4-x}\right) dx + \int_{0}^{4} 2\sqrt{4-x}dx = \frac{125}{6}$$

The problem is easier if you integrate with respect to y

$$\int_{-3}^{2} \left((4-y^2) - (y-2)\right) dy = \frac{125}{6}$$

□

Remark 70

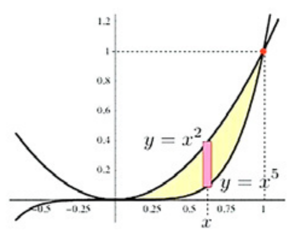
a) *You must be very careful to determine which function is above the other.*

b) *Keep in mind that although a definite integral can be positive, negative, or zero, the area of a region is always nonnegative.*

Example 71 *Find the area of the region bounded by the graphs of $y = x^2$ and $y = x^5$ two ways (integrating with respect to x and with respect to y).*

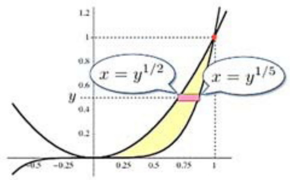
Solution:

a) Vertical slices:



$$A = \int_0^1 (x^2 - x^5)dx = \left(\frac{1}{3}x^3 - \frac{1}{6}x^6\right)|_0^1 =: \frac{1}{6}$$

b) Horizontal slices:



$$A = \int_0^1 (y^{1/5} - y^{1/2})dy = \left(\frac{5}{6}y^{6/5} - \frac{2}{3}y^{3/2}\right)|_0^1 = \frac{1}{6}$$

□

13.1.3 The Area under an Infinite Region

Sometimes by using limits, you can integrate a function defined on an unbounded interval. In the next example, we explore a so-called **improper integral** like

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx.$$

Example 72 *The area under the curve* $y = \dfrac{1}{x}$ *from* 1 *to* b *, a constant greater than* 1*, is given by*

$$Area = \int_1^b \frac{1}{x}dx = \ln b - \ln 1 = \ln b.$$

Hence, as b *tends to infinity, so does the area under this curve. On the other hand, the area under the curve* $y = \dfrac{1}{x^2}$ *from* 1 *to* b *is given by*

$$Area = \int_1^b \frac{1}{x^2}dx = \frac{b-1}{b}.$$

So as b *tends to infinity, the area approaches* 1*. We say that the improper integral equals* 1*, and we write*

$$\int_1^\infty \frac{1}{x^2}dx = 1$$

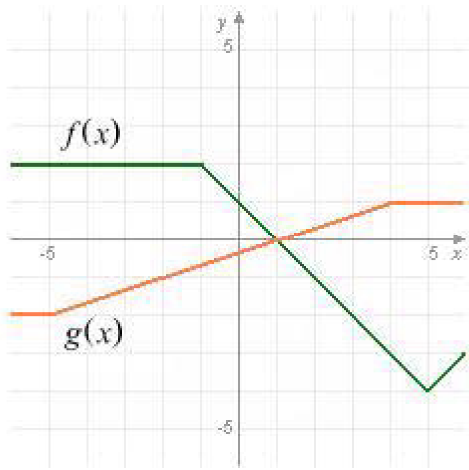
□

13.2 Exercises

Exercise 13.1 *Let* $F(x) = \int_0^x \frac{t-1}{1+t^2}dt$. *Find the critical points of* F *and at each critical point, determine if* F *has a local maximum, a local minimum, or neither a local maximum or a local minimum.*

Answer: $x = 1$ *is a critical point,* F *has local minimum at* $x = 1$.

Exercise 13.2 *Find the area between the graphs of* $f(x)$ *and* $g(x)$ *on the interval* $[-2, 5]$.



Answer: 16.

Exercise 13.3 *Find the area of the region bounded by the curves* $y = x^2 - 1$, $y - x + 2$, $x = 0$, *and* $x = 1$.

Answer: $\frac{13}{16}$.

Exercise 13.4 *Find the area of the region bounded by the curves* $y = -x^3 + 3$, $y = x$, $x = -1$, *and* $x = 1$.

Answer: 6.

Exercise 13.5 *Find the area of the region bounded by the curves* $y = \sqrt[3]{x-1}$ *and* $y = x - 1$.

Answer: $\frac{1}{2}$.

Exercise 13.6 *Set up and evaluate the definite integral that gives the area of the region bounded by the graph of* $f(x) = x^3$ *and its tangent line at* $(1, 1)$.

`Answer:` $\frac{27}{4}$.

Exercise 13.7 *Find $F'(x)$ if*

a) $F(x) = \int_0^{\sin x} \sqrt{t}dt$

b) $F(x) = \int_0^{\sin x} \left(1 - s^2\right) ds$

c) $F(x) = \int_x^{x^2} \dfrac{1}{s} ds$

`Answers:`

a) $\sqrt{\sin x}\,(\cos x)$

b) $\cos^3 x$

c) $\dfrac{1}{x^3} 3x^2 - \dfrac{1}{x}$

14

Natural logarithm (Exercises)

14.1 Practice Problems

14.1.1 Properties of the natural logarithmic function

Example 73 *Find*

a) $\frac{d}{dx}(x \ln x)$

b) $\frac{d}{dx} \ln(\cos x)$

Solution:

a) $\frac{d}{dx}(x \ln x) = (1) \ln x + x\frac{1}{x} = \ln x + 1$

b) $\frac{d}{dx} \ln(\cos x) = -\frac{1}{\cos x} \sin x = -\tan x.$

□

Example 74 *Find*

a) $\int \frac{x+1}{x^2} dx$

b) $\int \frac{x}{x^2+1} dx$

c) $\int \tan x dx$

d) $\int \frac{x}{\sqrt{x}\,(1+\sqrt{x})} dx$

Check (by differentiation) if the answers are correct.
Solution:

a) $\int \frac{x+1}{x^2} dx == \int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx = x \ln |x| - \frac{1}{x} + C$

b) $\int \frac{x}{x^2+1} dx = \frac{1}{2} \ln \left(x^2+1\right) + C$

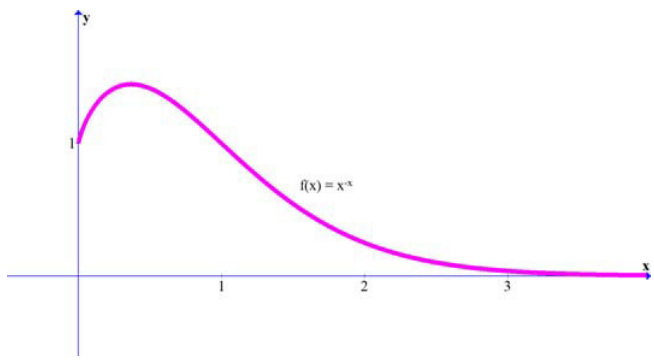
c) $\int \tan x dx = -\ln |\cos x| + C = \ln |\sec x| + C$

d) $\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})} dx = 2 \ln (x+\sqrt{x}) - \ln x + C$

14.1.2 Logarithmic differentiation

Example 75 *Find the derivative of the function*

$$f(x) = x^{-x}.$$



Solution 76

$$y = x^{-x}$$

$$\ln y = \ln x^{-x} = -x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \left((-1)\ln x + x\left(\frac{1}{x}\right)\right) = -(\ln x + 1)$$

(by the chain rule and by the product rule), so

$$\frac{dy}{dx} = -(\ln x + 1)x^{-x}$$

□

14.2 Exercises

15
The exponential function (Exercises)

15.1 Practice Problems

15.2 Exercises

Exercise 15.1 *Show that the function* $y = e^{ax} \sin bx$ *satisfies*

$$y'' - 2ay' + (a^2 + b^2)y = 0$$

for any real constanta a *and* b.

16
Inverse functions and inverse trig functions (Exercises)

16.1 Practice Problems

16.2 Exercises

Exercise 16.1 *Show that the function* $y = \arctan(x)$ *satisties*

$$y'' = -2 \sin y \cos^3 y$$

17

L'Hospital's rule and overview of limits (Exercises)

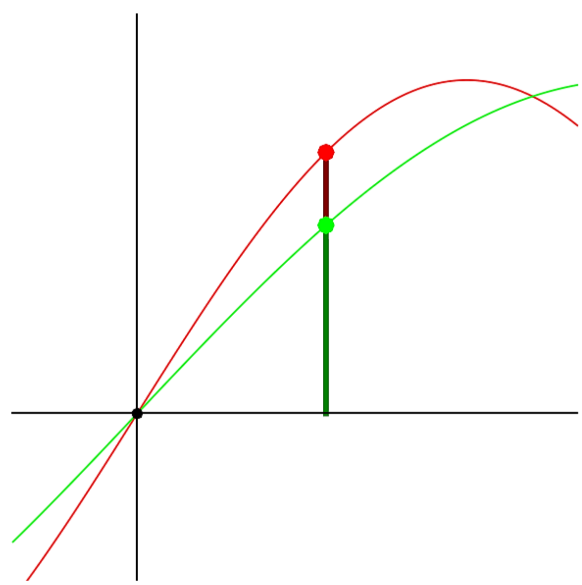
17.1 Practice problems

The L'Hopital's rule is a miracle procedure which solves all our worries about limits. The proof of the rule is comic in its simplicity. Especially after we will see how fantastically useful it is:

since $f(p) = g(p) = 0$ we have $Df(p) = f(p+h)/h$ and $Dg(p) = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) = 0$ the **quantum L'Hopital rule** holds:

$$\frac{f(p+h)}{g(p+h)} = \frac{\frac{f(p+h)}{h}}{\frac{g(p+h)}{h}} = \frac{\frac{f(p+h)-f(p)}{h}}{\frac{g(p+h)-g(p)}{h}}$$

Now take the limit $h \to 0$. Voilá!



Type A: $0/0$ case

Example 77 *Lets prove the **fundamental theorem of trigonometry** again:*

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

Why did we work so hard for this? We used the fundamental theorem to derive the derivatives for cos *and* sin *at all points. In order to apply L'Hopital, we had to know the derivative. Our work to establish the limit **was not in vain**.*
□

Example 78 *Sometimes, we have to administer a medicine twice. Let us find the limit* $\lim_{x\to 0} \frac{(1-\cos(x))}{x^2}$. *This limit had been pivotal to compute the derivatives of trigonometric functions.*

Solution: Differentiation gives

$$\lim_{x\to 0} \frac{\sin(x)}{2x}$$

Now apply L'Hopital again.

$$\lim_{x\to 0} \frac{\sin(x)}{2x} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2}$$

□

Example 79 *Find the limit*

$$\lim_{x\to 2} \frac{x^2-4x+4}{\sin^2(x-2)}$$

Solution: this is a case where $f(2) = f'(2) = g(2) = g'(2) = 0$ but $g''(0) = 2$. The limit is $f''(2)/g''(2) = 2/2 = 1$. □

Example 80 (Trouble?) *The limit* $\lim_{x\to\infty}(2x+\sin(x))/3x$ *is clearly* $2/3$ *since we can take the sum apart. Hopital gives* $lim_{x\to\infty}(2+\cos(x))/3$ *which has no limit. This is not trouble since Hopital applies only if the limit to right exists.*

Type A: ∞/∞ case

Example 81 *To find*

$$\lim_{x\to 0^+} \frac{\csc(x)}{1-\ln x}$$

notice that as $x \to 0^+$*, both the numerator and the denominator tend to* ∞*. Why? Well,* $\sin(x)$ *goes to* 0 *as* $x \to 0$*, so* $\csc(x)$ *blows up; and also* $\ln(x) \to -\infty$ *as* $x \to 0^+$*, so* $1 - \ln x \to \infty$*. Now use L'Hopital's Rule:*

$$\lim_{x\to 0^+} \frac{\csc(x)}{1-\ln x} = \lim_{x\to 0^+} \frac{-\csc(x)\cot(x)}{-\frac{1}{x}} = \lim_{x\to 0^+} x\csc(x)\cot(x).$$

To nd the limit, write it as

$$\lim_{x\to 0^+} \frac{x}{\sin(x)}\frac{1}{\tan(x)}.$$

We have

$$\lim_{x\to 0^+} \frac{x}{\sin(x)} = \lim_{x\to 0^+} \frac{1}{\frac{\sin(x)}{x}} = \frac{1}{1} = 1$$

but for the other factor we have

$$\lim_{x\to 0^+} \frac{1}{\tan(x)} = \infty$$

since $\tan(x) \to 0^+$ *as* $x \to 0^+$*. So we have proved that*

$$\lim_{x\to 0^+} \frac{\csc(x)}{1-\ln x} = \infty \qquad \square$$

Type $B1$ $(\infty - \infty)$

Example 82 *Find*

$$\lim_{x\to\infty} \sqrt{x+\ln(x)} - \sqrt{x}.$$

Solution: First note that as $x \to \infty$, both $\sqrt{x+\ln(x)}$ and $\sqrt{x}$ go to ∞; so we are in the $\infty - \infty$ case. There's no denominator, so let's make one by multiplying and dividing by the conjugate expression:

$$\lim_{x\to\infty} \sqrt{x+\ln(x)} - \sqrt{x} = \lim_{x\to\infty} \left(\sqrt{x+\ln(x)} - \sqrt{x}\right) \times \frac{\left(\sqrt{x+\ln(x)} + \sqrt{x}\right)}{\left(\sqrt{x+\ln(x)} + \sqrt{x}\right)}$$

Using the difference of squares formula $(a-b)(a+b)$, this becomes

$$\lim_{x\to\infty} \frac{\ln(x)}{\left(\sqrt{x+\ln(x)} + \sqrt{x}\right)}$$

Now we are in the ∞/∞ case of Type A, so we just differentiate top and bottom (using the chain rule on the bottom) to see that

$$\lim_{x\to\infty} \frac{\ln(x)}{\left(\sqrt{x+\ln(x)}+\sqrt{x}\right)} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}\frac{\frac{1}{x}+1}{\sqrt{x+\ln x}}+\frac{1}{2\sqrt{x}}}.$$

If you multiply the top and bottom of the fraction by x, you get

$$\lim_{x\to\infty} \frac{1}{\frac{1}{2}\frac{x+1}{\sqrt{x+\ln x}}+\frac{\sqrt{x}}{2}}.$$

We're almost done, but we do need to take a little look at what happens to the first fraction in the denominator as $\to\infty$:

$$\lim_{x\to\infty} \frac{1}{2}\frac{x+1}{\sqrt{x+\ln x}}.$$

This is also an ∞/∞ indeterminate form, so whack out another application of L'Hopital's Rule:

$$\lim_{x\to\infty} \frac{1}{2}\frac{x+1}{\sqrt{x+\ln x}} = \lim_{x\to\infty} \frac{1}{\frac{2}{2}\frac{\frac{1}{x}+1}{\sqrt{x+\ln x}}} = \lim_{x\to\infty} \frac{\sqrt{x+\ln x}}{\frac{1}{x}+1}$$

As $x\to\infty$, the denominator $\frac{1}{x}+1$ goes to 1 but the numerator $\sqrt{x+\ln x}$ goes to ∞. This means that

$$\lim_{x\to\infty} \frac{1}{2}\frac{x+1}{\sqrt{x+\ln x}} = \infty.$$

Returning to our original problem, we have already found that

$$\lim_{x\to\infty} \sqrt{x+\ln(x)}-\sqrt{x} = \lim_{x\to\infty} \frac{1}{\frac{1}{2}\frac{x+1}{\sqrt{x+\ln x}}+\frac{\sqrt{x}}{2}}.$$

Both fractions in the denominator go to ∞ as $x\to\infty$, so the limit is 0. □

Unfortunately, it's not always possible to use L'Hopital's Rule on type $B1$ limits. In fact, the only time it can actually work is when you're able to manipulate the original expression to be a ratio of two quantities, as in the above example.

Type $B2$ ($0\times\pm\infty$)

Example 83 *Find* $\lim_{x\to\infty} x\sin(1/x)$.

Solution: Write $y = 1/x$ then $sin(y)/y$. Now we have a limit, where the denominator and nominator both go to zero. Answer: 1.

Type C $(1^{\pm\infty}, 0^0$,or $\infty^0)$

Example 84 *Find*

$$\lim_{x\to 0^+} (\sin x)^x$$

Solution: Because direct substitution produces the indeterminate form 0^0 you can proceed as shown below. To begin, assume that the limit exists and is equal to

$y = \lim_{x\to 0^+} (\sin x)^x$	Indeterminate form 0^0
$\ln y = \ln\left(\lim_{x\to 0^+} (\sin x)^x\right)$	Take natural log of each side.
$= \left(\lim_{x\to 0^+} \ln (\sin x)^x\right)$	Continuity
$= \lim_{x\to 0^+} (x \ln (\sin x))$	Indeterminate form $0 \cdot (-\infty)$
$= \lim_{x\to 0^+} \left(\frac{\ln(\sin x)}{\frac{1}{x}}\right)$	Indeterminate form $\frac{-\infty}{\infty}$
$= \lim_{x\to 0^+} \left(\frac{\cot x}{-\frac{1}{x^2}}\right)$	L'Hopital's Rule
$= \lim_{x\to 0^+} \left(\frac{-x^2}{\tan x}\right)$	Indeterminate form $0/0$
$= \lim_{x\to 0^+} \left(\frac{-2x}{\sec^2 x}\right) = 0$	L'Hopital's Rule

Now, because $\ln y = 0$ you can conclude that $y = e^0 = 1$ and it follows that

$$\lim_{x\to 0^+} (\sin x)^x = 1. \qquad \square$$

Example 85 *Can L'Hopital's Rule be applied to* $\lim_{x\to 1} \dfrac{x^2+1}{2x+1}$?

Solution: The answer is no. The function does not have an indeterminate form because

$$\frac{x^2+1}{2x+1}|_1 = \frac{1^2+1}{2\cdot 1+1} = \frac{2}{3}.$$

However, the limit can be evaluated directly by substitution:

$$\lim_{x\to 1} \frac{x^2+1}{2x+1} = \frac{2}{3}.$$

An incorrect application of L'Hopital's Rule would lead to the following limit with a different value:

$$\lim_{x\to 1} \frac{2x}{2} = 1 \qquad \text{(not equal to original limit)}$$

$\square$

17.2 Exercises

Exercise 17.1 *What do you get if you apply L'Hopital to the limit*

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad ?$$

`Answer:` Differentiate both sides with respect to h. And then feel awsome!

Exercise 17.2 *Find the limit* $f(x)=\dfrac{(e^{x^2}-1)}{\sin(x^2)}$ *for* $x\to 0$.

`Answer: 1`

Exercise 17.3 *For the following functions, find the limits as* $x\to 0$*:*

a) $8x/\sin(x)$

b) $(e^x-1)/(e^{3x}-1)$

c) $\sin^2(3x)/\sin^2(5x)$

d) $\frac{\sin(x^2)}{\sin^2(x)}$

e) $\sin(\sin(x))/x$.

`Answer:` `a)`8 `b)` $\frac{1}{3}$ `c)`$\frac{9}{25}$ `d)`1 `e)`1.

Exercise 17.4 *For the following functions, find the limits as* $x\to 1$*:*

a) $(x^2-x-1)/(\cos(x-1)-1)$

b) $(e^x-e)/(e^{3x}-e^3)$

c) $(x-4)/(4x+\sin(x)+8)$

d) $\dfrac{x^{1000}-1}{x^{20}-1}$

e) $\dfrac{\tan^2(x-1)}{\cos(x-1)-1}$

Exercise 17.5 *Find the limit*

$$\lim_{x\to\infty}\frac{(x^2-x-1)}{\sqrt{x^4+1}}$$

`HINT:` first, then take the square root of the limit.

Exercise 17.6 *Find the limit*

$$\lim_{x\to\infty} \frac{e^x}{(1+e^x)}.$$

Exercise 17.7 *Use L'Hopital to compute the following limits at $x = 0$:*

a) $\lim_{x\to 0} \ln(5x)/\ln|x|$.

b) $\lim_{x\to 0}(\cos(x)x - \sin(x))/x^2$

c) $\lim_{x\to 0} \ln|\ln|1+x||/\ln|\ln|2+x||$.

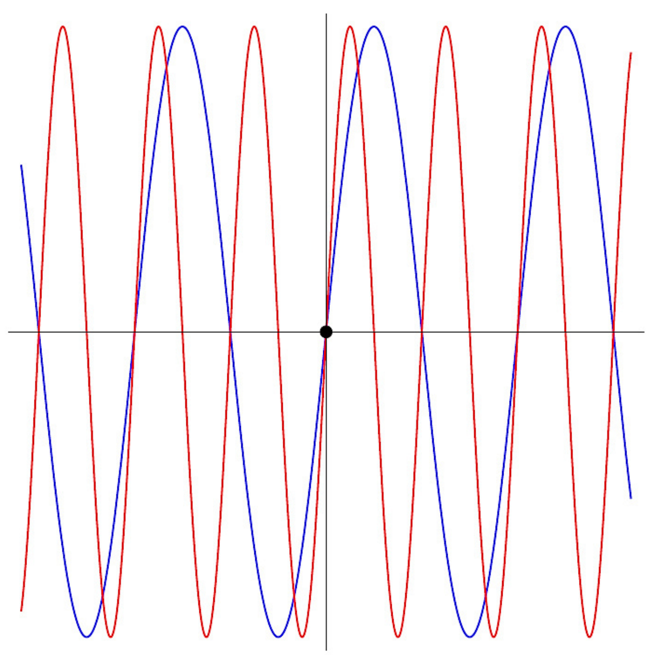
d) $\lim_{x\to 0} (1-e^x) / \left(x - x^3\right)$

e) $\lim_{x\to 0} \ln(1+3x)/x$

Exercise 17.8 *Apply L'Hopital's rule to get the limit of*

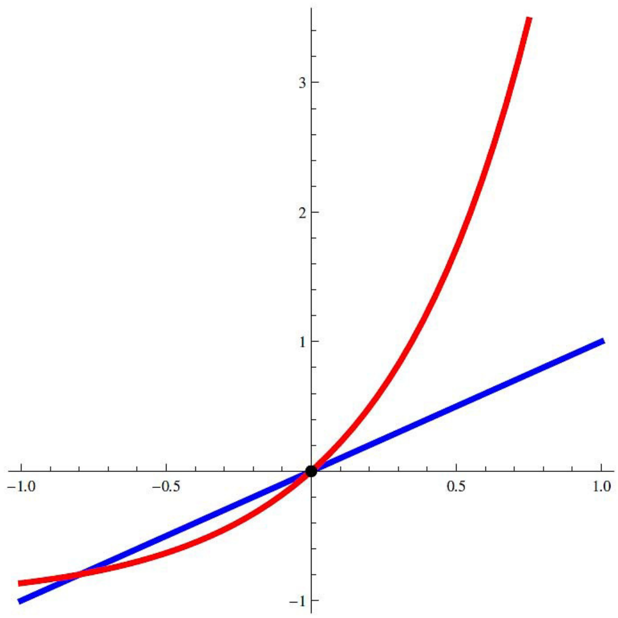
$$f(x) = \frac{\sin(200x)}{\sin(300x)}$$

for $x \to 0$.



Exercise 17.9 *What does L'Hopital's rule say about*

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}$$



18

Applications of Integration (Exercises)

18.1 Practice Problems

Example 86 *Find the volume of a right pyramid that has altitude h and square base of side a (see Figure 18.1).*

Solution: If, as shown in Figure 18.1, we introduce a coordinate line l along the axis of the pyramid, with origin 0 as a vertex, then cross sections by planes perpendicular to l are squares. If $A(x)$ is the cross sectional area determined by the plane that intersects the axis x units from 0, then

$$A(x) = (2y)^2 = 4y^2$$

while y is a distance indicated in the figure 18.1. By similar triangles

$$\frac{y}{x} = \frac{a/2}{h}, \qquad \text{or} \qquad y = \frac{ax}{2h}$$

and hence

$$A(x) = 4y^2 = 4\left(\frac{ax}{2h}\right)^2 = \frac{a^2}{h^2}x^2.$$

So,

$$V = \int_0^h \frac{a^2}{h^2}x^2 dx = \left(\frac{a^2}{h^2}\right)\frac{x^3}{3}|_0^h = \frac{1}{3}a^2 h$$

□

Example 87 *A solid has, as its base, the circular region in the xy-plane bounded by the graph of $x^2 + y^2 = a^2$, where $a > 0$. Find the volume of the solid if every cross section by the plane perpendicular to the x-axis is an equilateral triangle with one side in the base.*

Solution: A typical cross section by plane x units from the origin is illustrated if Figure 18.2. If the point $P(x, y)$ is on the circle, then the length of a side of the triangle is $2y$ and the altitude is $\sqrt{3}y$. Hence the area $A(x)$ of the pictured triangle is

$$A(x) = \frac{1}{2}(2y)\left(\sqrt{3}y\right) = \sqrt{3}y^2 = \sqrt{3}\left(a^2 - x^2\right).$$

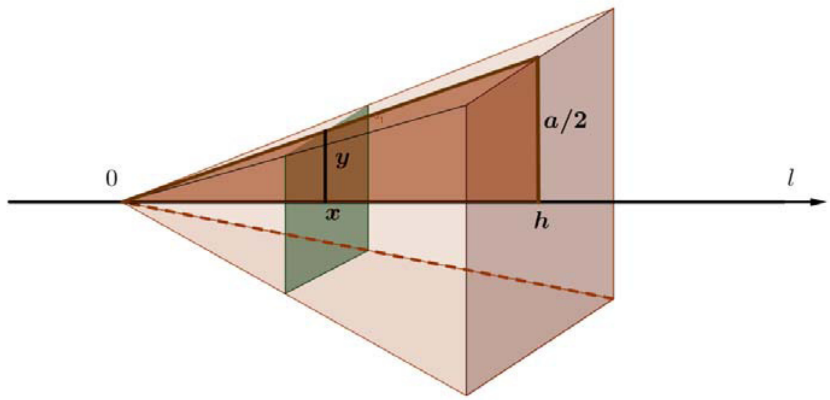


Fig. 18.1. Right pyramid that has altitude h

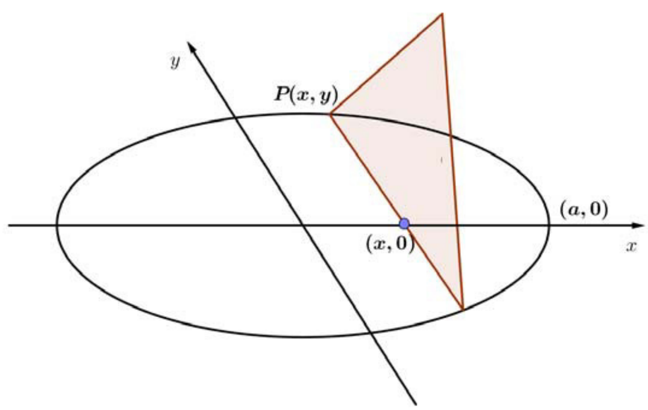


Fig. 18.2. A typical cross section of the solid from Example 87.

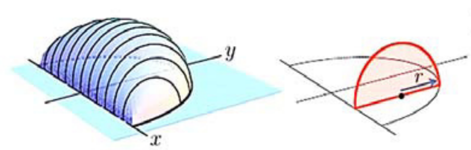
Now,

$$V = \int_{-a}^{a} \sqrt{3}\left(a^2 - x^2\right) dx = \frac{4}{3}\sqrt{3}a^3$$

(see also Example 89)□

Example 88 *A solid's base is the planar region in which* $0 \leq y \leq \sqrt{1-x^2}$ *and its vertical cross-sections parallel to the* y*-axis are semi-circles. Find the volume of the solid.*

Solution:



$$\begin{aligned} A(x) &= \frac{1}{2}\pi r^2 \\ &= \frac{1}{2}\pi\left(\frac{1}{2}\sqrt{1-x^2}\right)^2 \\ &= \frac{\pi}{8}\left(1-x^2\right). \end{aligned}$$

So

$$\begin{aligned} V &= \int_{-1}^{1} A(x)dx = \frac{\pi}{8}\int_{-1}^{1}\left(1-x^2\right)dx \\ &= \frac{\pi}{4}\int_{0}^{1}\left(1-x^2\right)dx = \frac{\pi}{4}(x - \frac{1}{3}x^3)|_0^1 \\ &= \frac{\pi}{4}\left(1 - \frac{1}{3} - 0\right) = \frac{\pi}{6} \end{aligned}$$

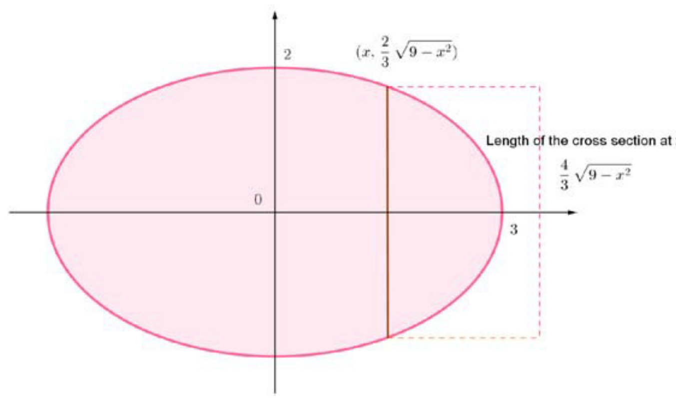
□

Example 89 *The base of a solid is the region bounded by the ellipse* $4x^2 + 9y^2 = 36$*. Find the volume of the solid given that cross sections perpendicular to the* x*-axis are*

a) *equilateral triangles,*

b) *squares.*

Solution:



a) The area of an equilateral triangle is $\frac{\sqrt{3}}{4}s^2$. We have $s = \frac{4}{3}\sqrt{9-x^2}$. thus, cross sectional areas are given by

$$A(x) = \frac{\sqrt{3}}{4}\left(\frac{4}{3}\sqrt{9-x^2}\right)^2 = 4\sqrt{3} - \frac{4}{9}\sqrt{3}x^2.$$

Therefore

$$V = \int_{-3}^{3} A(x)dx = \int_{-3}^{3}\left(4\sqrt{3} - \frac{4}{9}\sqrt{3}x^2\right)dx = 16\sqrt{3}$$

b) The area of a square is s^2. We have $s = \frac{4}{3}\sqrt{9-x^2}$. Thus, cross sectional areas are given by

$$A(x) = \left(\frac{4}{3}\sqrt{9-x^2}\right)^2 = 16 - \frac{16}{9}x^2.$$

Therefore

$$V = \int_{-3}^{3} A(x)dx = \int_{-3}^{3}\left(16 - \frac{16}{9}x^2\right)dx = 64.$$

□

Example 90 *We can generate a circular cone of base radius r and height h by revolving about the x-axis the region below the graph of*

$$f(x) = \frac{r}{h}x, \quad 0 \le x \le h$$

(see Figure 18.3). Find volume of that cone by Disc Method.

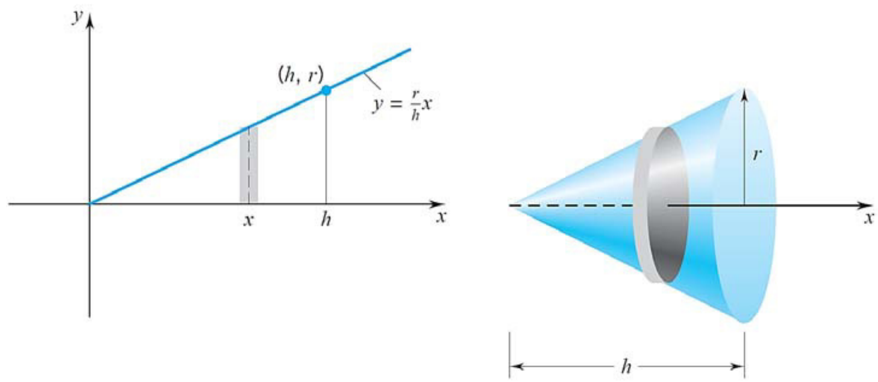


Fig. 18.3. Circular cone of base radius r and height h

Solution: We have

$$V = \int_0^h \pi \left(\frac{r}{h}x\right)^2 dx = \frac{\pi r^2}{h^2}\left(\frac{x^3}{3}\right)|_0^h = \frac{1}{3}\pi h r^2.$$

□

Example 91 *A manufacturer drills a hole through the center of a metal sphere of radius* 5 *inches, as shown in Figure 18.4. The hole has a radius of* 3 *inches. What is the volume of the resulting metal ring?*

Solution: You can imagine the ring to be generated by a segment of the circle whose equation $x^2 + y^2 = 25$ is as shown in Figure 18.5.

Because the radius of the hole is 3 inches, you can let $y = 3$ and solve the equation $x^2 + y^2 = 25$ to determine that the limits of integration are $x = \pm 4$. So, the inner and outer radii are $r(x) = 3$ and $R(x) = \sqrt{25 - x^2}$ and the volume is given by

$$\begin{aligned}
V &= \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right) dx \\
&= \pi \int_{-4}^4 \left(\left[\sqrt{25 - x^2}\right]^2 - [3]^2\right) dx \\
&= \pi \int_{-4}^4 (16 - x^2) dx \\
&= \pi \left[16x - \frac{x^3}{3}\right]_{-4}^4 \\
&= \frac{256}{3}\pi \quad \text{cubic inches.}
\end{aligned}$$

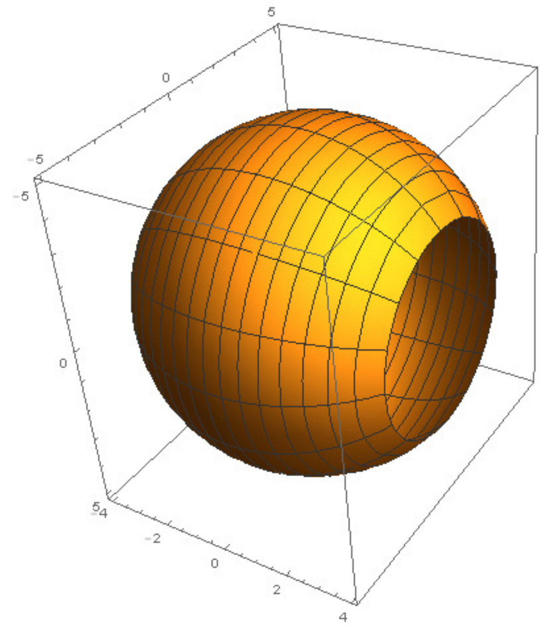


Fig. 18.4. Solid of revolution

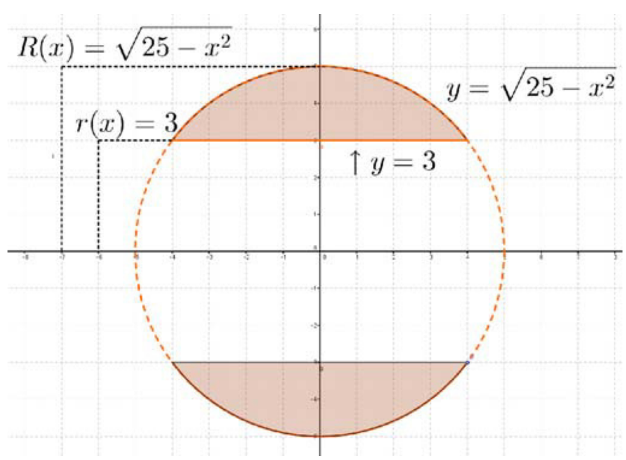


Fig. 18.5. Plane region.

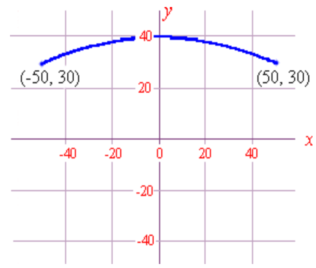


Fig. 18.6. Side of the barrel.

□

Example 92 *A wine cask has a radius at the top of* 30 *cm. and a radius at the middle of* 40 *cm. The height of the cask is* 1 *m. What is the volume of the cask (in L), assuming that the shape of the sides is parabolic?*

Solution: We will lay the cask on its side to make the algebra easier (see Fig. 18.6) We need to find the equation of a parabola with vertex at $(0, 40)$ and passing through $(50, 30)$. We use the formula:

$$(x-h)^2 = 4a(y-k)$$

Now (h, k) is $(0, 40)$ so we have:

$$x^2 = 4a(y-40)$$

and the parabola passes through $(50, 30)$, so

$$(50)^2 = 4a(30-40)$$

and

$$2500 = 4a(-10).$$

This gives

$$4a = -250$$

So the equation of the side of the barrel is

$$x^2 = -250(y-40),$$

that is,

$$y = -\frac{x^2}{250} + 40.$$

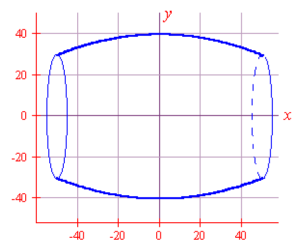


Fig. 18.7. The barrel as a rotated parabola.

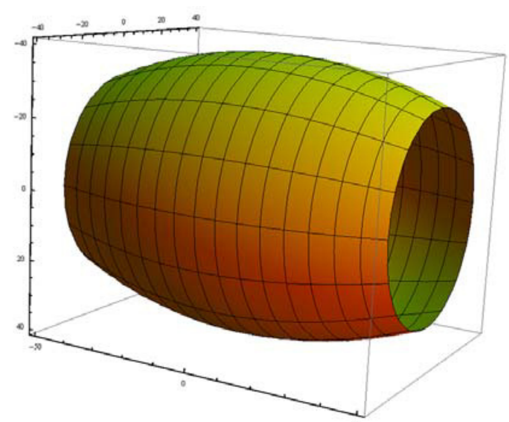


Fig. 18.8.

Fig. 18.9. Watermelons.

We need to find the volume of the cask which is generated when we rotate this parabola between $x = -50$ and $x = 50$ around the x-axis (see Fig.**??**).

$$\begin{aligned}
\text{Vol} &= \pi \int_a^b y^2 dx \\
&= \pi \int_{-50}^{50} \left(-\frac{x^2}{250} + 40\right)^2 dx \\
&= \pi \int_{-50}^{50} \left(\frac{x^4}{62\,500} - \frac{8x^2}{25} + 1600\right) dx \\
&= \pi \left[\frac{x^5}{312\,500} - \frac{8x^3}{75} + 1600x\right]_{-50}^{50} \\
&= 2\pi \left(\frac{(50)^5}{312\,500} - \frac{8\,(50)^3}{75} + 1600\,(50)\right) \\
&= 4.251\,6 \times 10^5
\end{aligned}$$

So the wine cask will hold 425.16 L. □

Example 93 *A watermelon has an ellipsoidal shape with major axis* 28 *cm. and minor axis* 25 *cm. (see Fig.18.9). Find its volume.*

Before calculus, one way of approximating the volume would be to slice the watermelon (say in 2 cm. thick slices) and add up the volumes of each slice using $V = 2\pi rh$. Interestingly, Archimedes (the one who famously jumped out of his bath and ran down the street shouting "Eureka! I've got it") used this approach to find volumes of spheres around 200 BC. The technique was almost forgotten until the early 1700's when calculus was developed by Newton and Leibniz.

We see how to do the problem using both approaches.

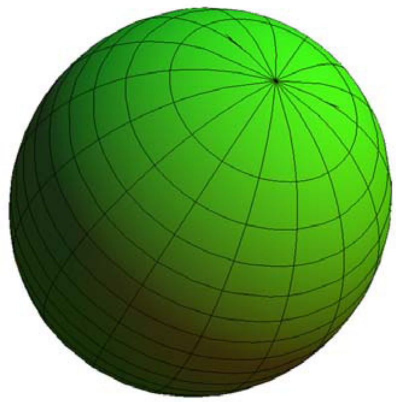


Fig. 18.10. Slices of a particular watermelon.

Historical Approach: Because the melon is symmetrical, we can work out the volume of one half of the melon, and then double our answer. The radii for the slices for one half of a particular watermelon are found from measurement to be:

$$0,\ 6.4,\ 8.7\ , 10.3\ , 11.3\ , 12.0\ , 12.4\ , 12.5.$$

The approximate volume for one half of the melon using slices 2 cm. thick would be:

$$\begin{aligned} V_{half} &= \pi \times [6.4^2 + 8.7^2 + 10.3^2 + 11.3^2 + 12.0^2 + 12.4^2 + 12.5^2] \times 2 \\ &= \pi \times 804.44 \times 2 = 5054.4 \end{aligned}$$

So the volume for the whole watermelon is about $5054.4 \times 2 = 10109$ cm^3 = 10.1 L.

"Exact" Volume (using Integration): We are told the melon is an ellipsoid. We need to find the equation of the cross-sectional ellipse with major axis 28 cm. and minor axis 25 cm. We use the formula

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a is half the length of the major axis and b is half the length of the minor axis. For the volume formula, we will need the expression for y^2 and it

is easier to solve for this now (before substituting our a and b).

$$\begin{aligned}
\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1 \\
b^2x^2+a^2y^2 &= a^2b^2 \\
a^2y^2 &= a^2b^2-b^2x^2=b^2(a^2-x^2)
\end{aligned}$$

so

$$y^2=\frac{b^2}{a^2}(a^2-x^2).$$

Since $a=14$ and $b=12.5$, we have:

$$y^2=\frac{12.5^2}{14^2}(14^2-x^2)=156.25-0.797\,19x^2.$$

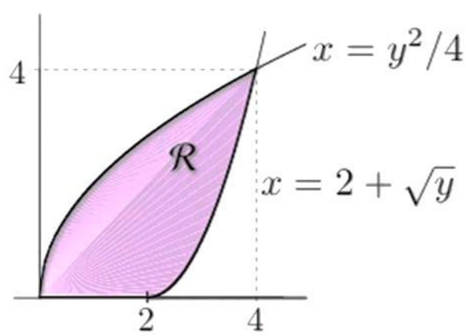
NOTE: The a and b that we are using for the ellipse formula are not the same a and b we use in the integration step. They are completely different parts of the problem.

Using this, we can now find the volume using integration. (Once again we find the volume for half and then double it at the end).

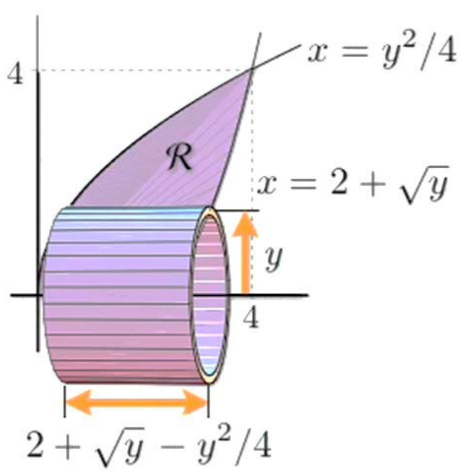
$$\begin{aligned}
V_{half} &= \pi\int_0^{14} y^2dx \\
&= \pi\int_0^{14}\left(156.25-0.797\,19x^2\right)dx \\
&= \pi\left[156.25x-0.265\,73x^3\right]_0^{14} \\
&= 1458.3\pi=4580.65 \text{ cm}^3
\end{aligned}$$

So the watermelon's total volume is $2\times 4580.65=9161$ cm^3 or 9.161 L. This is about the same as what we got by slicing the watermelon and adding the volume of the slices. □

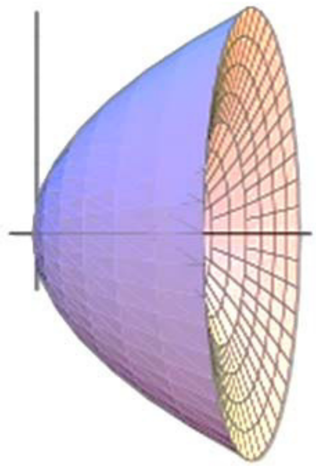
Example 94 *Let $\mathcal{R}$ be the region in which $y^2/4 \le x \le 2 + \sqrt{y}$. Find the volume of the solid obtained by revolving $\mathcal{R}$ about the x-axis.*



Solution:



$$
\begin{aligned}
dV &= \underset{circumference}{2\pi y \cdot} \quad \underset{height}{\left(2 + \sqrt{y} - y^2/4\right)} \quad \underset{thickness}{dy} \\
&= 2\pi(2y + y^{3/2} - y^3/4)dy
\end{aligned}
$$



$$\begin{aligned} V &= 2\pi \int_0^4 (2y + y^{3/2} - y^3/4) dy \\ &= 2\pi \left(y^2 - \frac{1}{16} y^4 + \frac{2}{5} y^{\frac{5}{2}} \right) |_0^4 = \frac{128}{5}\pi \end{aligned}$$

□

Example 95 *Find the volume of the solid of revolution formed when the region bounded between* $x = -2.5$*,* $x = 4.4$*,* $y = \cos x + 2$ *and* $y = 0$ *is revolved vertically around the* x*-axis.*

Solution:

$$V = \pi \int_{-2.5}^{4.4} (\cos(x) + 2)^2 dx = \pi \left(\frac{9}{2} x + 4 \sin x + \frac{1}{4} \sin 2x \right) |_{-2.5}^{4.4} = 29.544\pi$$

□

Example 96 *Sketch the region bounded by* $y = \sqrt{x}$*,* $x = 4$*, and* $y = 0$*. Use the shell method to find the volume of the solid generated by revolving the region about the* y*-axis.*

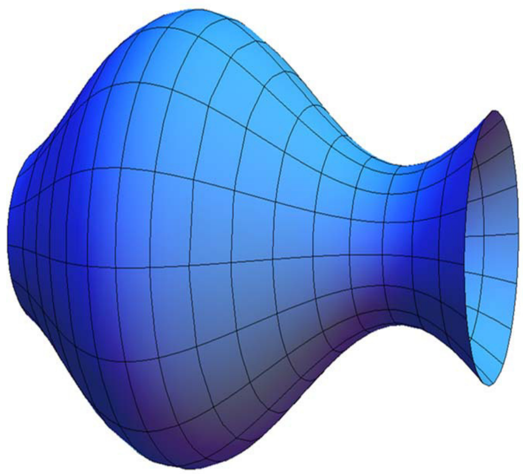
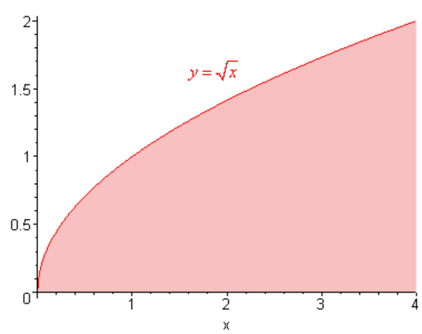


Fig. 18.11. Solid of revolution defined in the Example 95

Solution 97



$$V=\int_0^4 2\pi x\sqrt{x}dx=2\pi\int_0^4 x^{3/2}dx=2\pi\left(\frac{2}{5}x^{5/2}\right)|_0^4=\frac{128}{5}\pi.$$

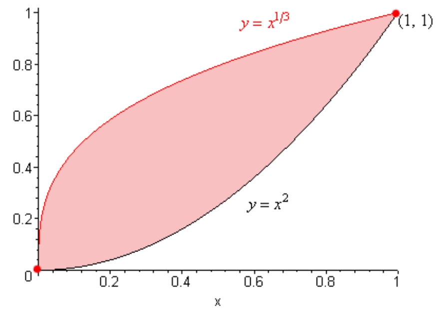
The same result can be obtained by the disc method as follows:

$$V=\int_0^2 \pi\left((4)^2-\left(y^2\right)^2\right)dy=\frac{128}{5}\pi.$$

□

Example 98 *Sketch the region bounded by $y = x^2$ and $y = x^{1/3}$. Use the shell method to find the volume of the solid generated by revolving the region about the y-axis.*

Solution:



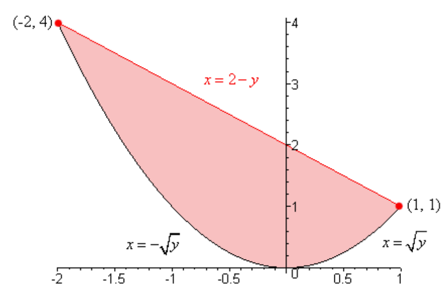
The points of intersection of the curves $y = x^2$ and $y = x^{1/3}$ are $(0,0)$ and $(1,1)$.

$$\begin{aligned} V &= \int_0^1 2\pi x \left[x^{1/3} - x^2\right] dx = 2\pi \int_0^1 x \left[x^{1/3} - x^2\right] dx \\ &= 2\pi \left(\frac{3}{7}x^{7/3} - \frac{1}{4}x^4\right)|_0^1 = \frac{5}{14}\pi. \end{aligned}$$

□

Example 99 *Sketch the region bounded by $y = x^2$ and $y = 2 - x$. Use the shell method to find the volume of the solid generated by revolving the region about the x-axis.*

Solution:



We first find the points of intersection of the curves $y = x^2$ and $y = 2 - x$

$$\begin{aligned} x^2 &= 2 - x \\ (x+1)(x+2) &= 0 \\ x &= 1 \quad \text{or} \quad x = -2. \end{aligned}$$

The curves intersect at $(1,1)$ and $(-2,4)$.

$$\begin{aligned} V &= \int_0^1 2\pi y\,(\sqrt{y}-(-\sqrt{y}))\,dy + \int_1^4 2\pi y\,((2-y)-(-\sqrt{y}))\,dy \\ & \left(\frac{8}{5}\pi y^{\frac{5}{2}}\right)|_0^1 + \left(2\pi y^2 - \frac{2}{3}\pi y^3 + \frac{4}{5}\pi y^{\frac{5}{2}}\right)|_1^4 \\ &= \frac{8}{5}\pi + \frac{64}{5}\pi = \frac{72}{5}\pi. \end{aligned}$$

The volume of the same solid can be found by the washer method as follows:

$$\int_{-2}^1 \pi((2-x)^2 - (x^2)^2)dx = \frac{72}{5}\pi.$$

□

18.2 Exercises

Exercise 18.1 *Find the volume of the solid of revolution formed when the region bounded between* $x=1$, $x=3$, $y=x^2$ *and* $y=0$ *is revolved vertically around the* x*-axis.*

`Answer:` $\frac{243\pi}{5}$

Exercise 18.2 *Find the volume of the solid of revolution formed when the region below the curve* $y=x^2-1$, *between* $x=1$ *and* $x=3$, *and above* $y=0$ *is revolved around the* x*-axis.*

`Answer:` $\frac{496}{15}\pi$

Exercise 18.3 *Find the volume of the solid of revolution formed when the region bounded between* $x=0$, $x=\pi$, $y=\sin(x)$ *and* $y=0$.

`Answer:` $\frac{1}{2}\pi^2$

Exercise 18.4 *Find the volume of the solid of revolution formed when the region bounded between* $x=0$, $x=pi/2$, $y=\cos(x)$ *and* $y=0$.

`Answer:` $\frac{1}{4}\pi^2$

Exercise 18.5 *Find the volume of the solid of revolution formed when the region bounded between* $x=\pi/4$, $x=\pi/2$, $y=\sin(x)$ *and* $y=1/2$.

`Answer:` $\pi\left(\frac{1}{16}\pi+\frac{1}{4}\right)$

Exercise 18.6 *Find the volume of the solid of revolution formed when the region below the curve $y = e^x/2$, between $x = 3$ and $x = 5$, and above $y = 2$ is revolved around the x-axis.*

`Answer:` $-\pi\left(e^3 - e^5 + 8\right)$

Exercise 18.7 *Find the volume of the solid of revolution formed when the region below the curve $y = 4\sqrt{x}$, between $x = 4$ and $x = 9$, and above $y = 4$ is revolved around the line $y = 2$.*

`Answer:` $\frac{952}{3}\pi$

Exercise 18.8 *Find the volume of the solid of revolution formed when the region bounded between $y = x + 5$ and $y = x^2 + 3$ is revolved vertically around the line $y = 2$.*

`HINT`: Find the intersection points first.
`Answer:` $\frac{117}{5}\pi$

Exercise 18.9 *Find the volume of the solid of revolution formed when the region bounded between $y = x$ and $y = x^2$ is revolved horizontally around the y-axis.*

`Answer:` $\frac{1}{6}\pi$

Exercise 18.10 *Find the volume of the solid of revolution formed when the region bounded between $y = 2x - 2$ and $y = (x - 1)^2$ is revolved horizontally around the y-axis.*

`Answer:` $\frac{16}{3}\pi$

Exercise 18.11 *Find the volume of the solid of revolution formed when the region bounded between $y = 2x - 6$ and $y = (x - 3)^2$ is revolved horizontally around the line $x = 1$.*

`Answer:` 8π

Exercise 18.12 *Find the volume of the solid of revolution formed when the region bounded between $y = x + 2$ and $y = x^2$ is revolved vertically around the line $y = -2$.*

`Answer:` $\frac{162}{5}\pi$

Exercise 18.13 *Find the volume of the solid of revolution formed when the region bounded between $y = x$ and $y = x^2$ is revolved horizontally around the line $x = -1$.*

`Answer:` $\frac{1}{2}\pi$

Exercise 18.14 *Find the volume of the solid of revolution formed when the region bounded between $y = 2x$ and $y = x^2$ is revolved horizontally around the line $x = -3$.*

`Answer:` $\frac{32}{3}\pi$

Exercise 18.15 *Sketch the region bounded by $y = x$ and $y = 5$ and $x = 0$. Use the shell method to find the volume of the solid generated by revolving the region about the y-axis.*

`Answer:` $\frac{250}{3}\pi$

Exercise 18.16 *Sketch the region bounded by $y = \sqrt{x}$, $x = 0$ and $y = 1$. Use the shell method to find the volume of the solid generated by revolving the region about line $y = 2$.*

`Answer:` $\frac{5}{6}\pi$

Exercise 18.17 *Consider a cap of thickness h that has been sliced from a sphere of radius r (see figure). Verify, that the volume of the cap is*

$$\pi h^2(3r - h)/3$$

using:

a) *the washer method,*

b) *the shell method,*

c) *general slicing method.*

Exercise 18.18 *The following integrals match the volumes of solids (Fig. 18.13). Each integral matches exactly one solid.*

Integral	*Enter* 1-6
$\int_0^1 \pi x^2 dx$	
$\int_0^1 \pi dx$	
$\int_0^1 \pi(1 - x^2)dx$	
$\int_0^1 \pi \sin^2(\pi x)\, dx$	
$\int_0^1 \pi(1 + x)^2 dx$	
$\int_0^1 \pi \cos^2(\pi x)\, dx$	

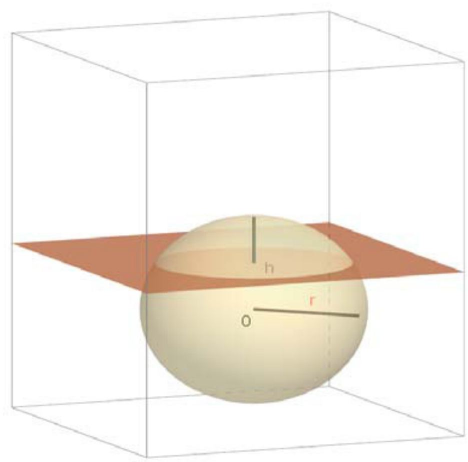


Fig. 18.12.

Exercise 18.19 *Match the volumes of solids (Figure 18.14).*

Integral	*Enter* 1-6
$\int_0^1 \pi z^4 dz$	
$\int_0^1 \pi z dz$	
$\int_0^1 \pi(4+\sin(4z))dz$	
$\int_0^1 \pi e^{-4z^2} dz$	
$\int_0^1 \pi z^2 dz$	
$\int_0^1 (1-z)^2 dz$	

Exercise 18.20 *The kiss is a solid of revolution for which the radius at height z is*

$$z^2\sqrt{1-z}$$

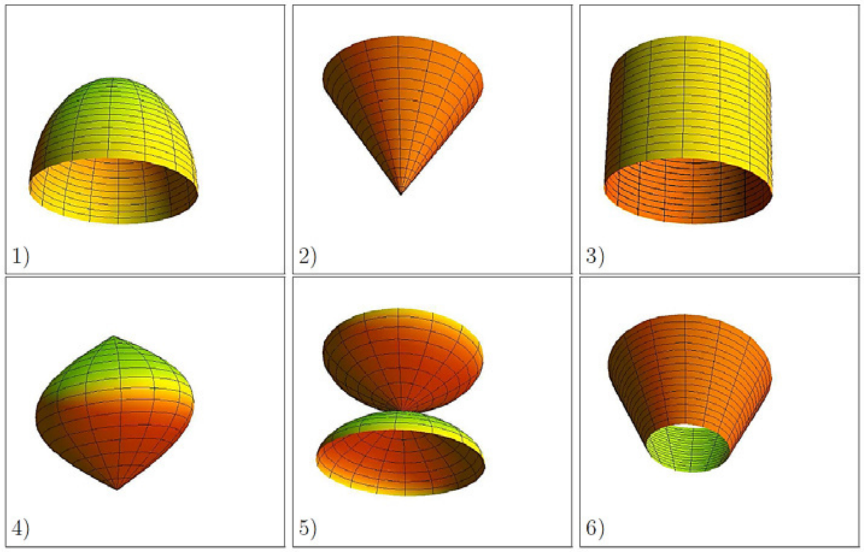


Fig. 18.13. Solids for exercise 18.18

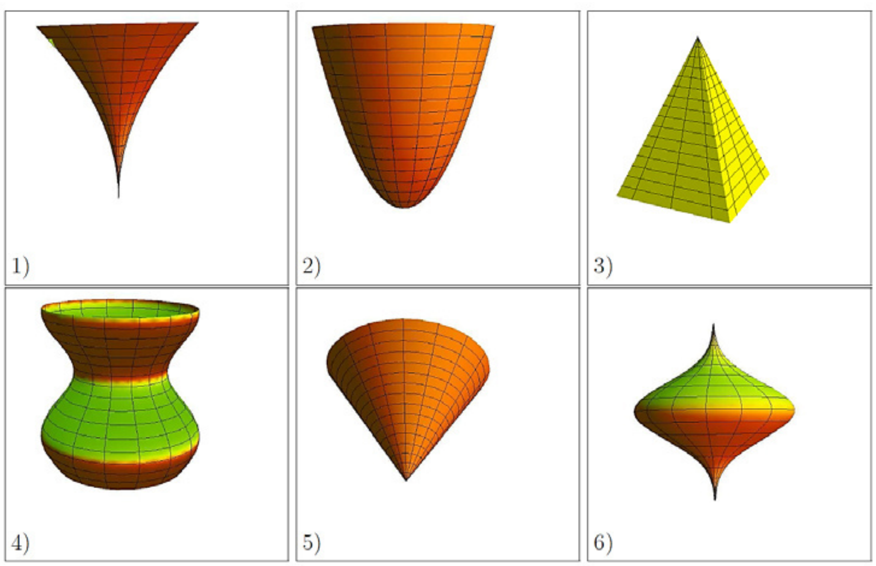
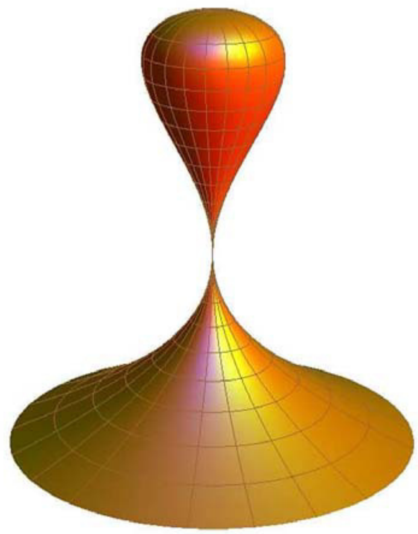


Fig. 18.14. Solids for exercise 18.19

and where $-1 \leq z \leq 1$*. What is the volume of this solid? The name "kiss" is the official name for this quartic surface.*



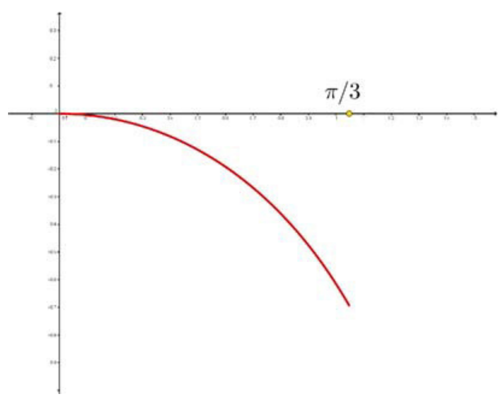
19

Arc Length and Surface Area (Exercises)

19.1 Practice Problems

Example 100 *Find the arc length of the graph of the function* $f(x) = \ln(\cos x)$ *over the interval* $1 \leq x \leq \dfrac{\pi}{2}$.

Solution:



$$f'(x) = -\frac{\sin x}{\cos x} = \tan x.$$

$$1 + \left(f'(x)\right)^2 = \frac{1}{\cos^2 x}\sin^2 x + 1 = \frac{1}{\cos^2 x}.$$

$$\begin{aligned}
s &= \int_a^b \sqrt{1 + (f'(x))^2}dx = \\
\int_0^{\pi/3} \frac{1}{\cos x}dx &= \left(\frac{1}{2}\ln(2\sin x + 2) - \frac{1}{2}\ln(2 - 2\sin x)\right)|_0^{\pi/3} \\
&= \frac{1}{2}\ln\left(\sqrt{3} + 2\right) - \frac{1}{2}\ln\left(2 - \sqrt{3}\right) \approx 1.3170
\end{aligned}$$

□

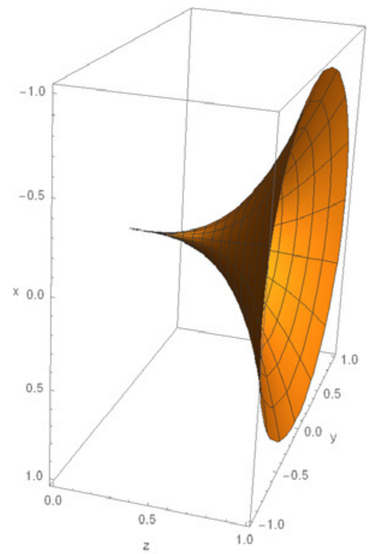


Fig. 19.1.

Example 101 *Find the area of the surface obtained by revolving the graph of* x^3 *on* $[0,1]$ *about the* x *axis.*

Solution: We find that $f'(x) = 3x^2$ and $\sqrt{1 + (f'(x))^2} = \sqrt{1 + 9x^4}$, so

$$\begin{aligned} A &= 2\pi \int_0^1 \sqrt{1+9x^4}x^3 dx \\ &= \frac{\pi}{2} \int_0^1 \sqrt{1+9u}du \qquad (u = x^4,\ du = 4x^3 dx) \\ &= \frac{\pi}{18} \int_0^9 \sqrt{1+v}dv \qquad (u = \frac{1}{9}v,\ du = \frac{1}{9}dv) \\ &= \frac{1}{18}\pi \left(\frac{20}{3}\sqrt{10} - \frac{2}{3} \right) \approx 3.5631 \end{aligned}$$

□

Example 102 (Surface area of a spherical cap) *A spherical cap is produced when a sphere of radius* r *is sliced by a horizontal plane that is a vertical distance* h *below the north pole of the sphere, where* $0 \le h \le 2a$ *(Figure 19.2). We take the spherical cap to be that part of the sphere above the plane, so that* h *is the depth of the cap. Show that the area of a spherical cap of depth* h *cut from a sphere of radius* r *is* $2\pi r h$*.*

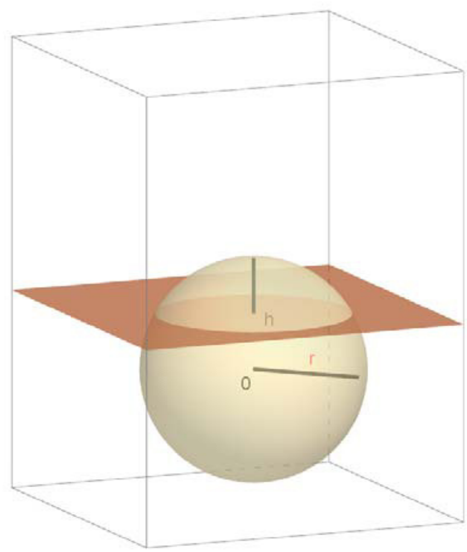


Fig. 19.2. A spherical cap

Solution: To generate the spherical surface, we revolve the curve $f(x) = \sqrt{r^2 - x^2}$ on the interval $[-r, r]$ about the x-axis. The spherical cap of height h corresponds to that part of the sphere on the interval $[r-h, r]$, for $0 \leq h \leq 2r$. Noting that $f'(x) = -x\left(r^2 - x^2\right)^{-1/2}$, the surface area of the spherical cap of height h is

$$\begin{aligned} S &= \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}dx \\ &= 2\pi \int_{r-h}^{r} \sqrt{r^2 - x^2}\sqrt{1 + \left(-x\left(r^2 - x^2\right)^{-1/2}\right)^2}dx \\ &= 2\pi \int_{r-h}^{r} \sqrt{\frac{x^2}{r^2 - x^2} + 1}\sqrt{r^2 - x^2}dx \\ &= 2\pi \int_{r-h}^{r} r dx = 2\pi r h. \end{aligned}$$

□

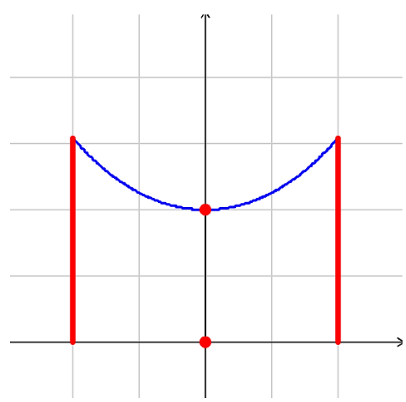
Remark 103 *Notice that f is not differentiable at $\pm r$. Nevertheless, in this case, the surface area integral can be evaluated using methods you know.*

19.2 Exercises

Exercise 19.1 *A cable is to be hung between two poles of equal height that are 20m. apart. Suppose that the cable takes the shape of*

$$f(x) = 5\left(e^{x/10} + e^{-x/10}\right).$$

Find the length of the cable.



`Answer:` ≈ 23.504

20

Techniques of integration, part one (Exercises)

20.1 Basic Integration Formulas

$$
\begin{array}{rl}
\mathbf{1.} & \int kdu = ku + C \\
\mathbf{2.} & \int u^p du = \frac{u^{p+1}}{p+1} + C \quad p \neq 1 \\
\mathbf{3.} & \int \cos au du = \frac{1}{a} \sin au + C \\
\mathbf{4.} & \int \sin au du = -\frac{1}{a} \cos au + C \\
\mathbf{5.} & \int \sec^2 au du = \frac{1}{a} \tan au + C \\
\mathbf{6.} & \int \csc^2 au du = -\frac{1}{a} \cot au + C \\
\mathbf{7.} & \int \sec au \tan au du = \frac{1}{a} \sec au + C \\
\mathbf{8.} & \int \csc au \cot au du = -\frac{1}{a} \csc au + C \\
\mathbf{9.} & \int e^{au} du = \frac{1}{a} e^{au} + C \\
\mathbf{10.} & \int \frac{1}{u} du = \ln |u| + C \\
\mathbf{11.} & \int \frac{1}{\sqrt{a^2-u^2}} du = \arcsin \frac{u}{a} + C \\
\mathbf{12.} & \int \frac{1}{a^2+u^2} du = \frac{1}{a} \arctan \frac{u}{a} + C
\end{array}
\tag{20.1}
$$

20.2 Practice Problems

20.2.1 Integration by substitution

The first principle of substitution is:

$$\boxed{\begin{array}{l}\text{When you have a quantity to a power,}\\ \text{let } u \text{ equal that quantity and try to get } du.\end{array}} \tag{20.2}$$

Warning: $u =$ the quantity, not the quantity to the power.

Example 104 *Find*

$$\int x^2 \left(2x^3+5\right)^2 dx.$$

Solution: Here we have quantity to a power. We can try to use the Power Rule.

$$\int x^2 \left(2x^3+5\right)^2 dx = \frac{1}{6} \int \left(2x^3+5\right)^2 \frac{d}{dx} \left(2x^3+5\right) dx.$$

Let

$$u = 2x^3 + 5$$

then we need to compute

$$\int u^2 du = \frac{u^3}{3} + C = \frac{1}{18}\left(2x^3+5\right)^3 + C.$$

This is the answer, which we can check by differentiating:

$$\frac{d}{dx}\left(\frac{1}{18}\left(2x^3+5\right)^3 + C\right) = x^2\left(2x^3+5\right)^2$$

which is the original integrand, verifying that the answer is correct. □

Example 105 *Find*

$$\int \frac{2\cos 3x}{(5+\sin 3x)^4}dx.$$

Solution: Again we have a quantity to a power, so we let u equal the quantity and try to get du

$$u = 5 + \sin 3x, \qquad du = 3\cos 3x dx.$$

To avoid unnecessary confusion, we move the "2" outside the integral and rewrite this as

$$\begin{aligned} & \frac{2}{3}\int \frac{1}{(5+\sin 3x)^4}\frac{d}{dx}(5+\sin 3x)\,dx \\ = & \frac{2}{3}\int \frac{1}{u^4}du = -\frac{2}{9}u^{-3} + C \\ = & \frac{-2}{9\,(5+\sin 3x)^3} + C. \end{aligned}$$

This is the answer which you check by differentiating:

$$\frac{d}{dx}\left(\frac{-2}{9\,(5+\sin 3x)^3} + C\right) = 2\frac{\cos 3x}{(\sin 3x+5)^4}.$$

□

Example 106 *Find*

$$\int x^3 \sin(5x^4+6)dx$$

Here, we do not have a quantity to a power, so the first principle of substitution does not apply. Hence, we turn to the second principle of substitution.

$$\boxed{\begin{array}{l}\text{When you do not have a quantity to a power,}\\ \text{but you do have a trig function of a quantity}\\ \text{let } u \text{ equal that quantity and try to get } du.\end{array}} \tag{20.3}$$

Later, this same principle will equally apply to other kinds of functions, such as exponential, logarithmic, inverse trig, etc.

Solution: Let $u = 5x^4 + 6$ then $du = 20x^3 dx$. Then

$$\begin{aligned} \int x^3 \sin(5x^4+6)dx &= \frac{1}{20}\int \sin u du \\ &= -\frac{1}{20}\cos u + C \\ &= -\frac{1}{20}\cos\left(5x^4+6\right) + C \end{aligned}$$

This is the answer, which we can check by differentiating:

$$\frac{d}{dx}\left(-\frac{1}{20}\cos\left(5x^4+6\right)+C\right) = x^3 \sin\left(5x^4+6\right).$$

□

Example 107 *Find*

$$\int \frac{9\sec^2(3\sqrt{x}+5)}{\sqrt{x}}dx$$

Solution: We recognize that we have a formula for $\int$ in (20.1), so we move toward putting the problem into this form. Let

$$u = 3\sqrt{x}+5, \qquad du = \frac{3}{2}\frac{1}{\sqrt{x}}dx.$$

We first move the "9" outside of the integral and move the $\frac{1}{\sqrt{x}}$ next to the dx

$$9\int \sec^2(3\sqrt{x}+5)\frac{1}{\sqrt{x}}dx.$$

Next multiply by $\frac{2}{3}\cdot\frac{3}{2}$, move the unwanted $\frac{2}{3}$ outside of the integral, simplify and substitute.

$$\begin{aligned}
9\int \sec^2(3\sqrt{x}+5)\frac{1}{\sqrt{x}}dx &= 9\int \sec^2(3\sqrt{x}+5)\cdot\frac{3}{2}\frac{1}{\sqrt{x}}dx \\
&= 9\frac{2}{3}\int \sec^2(3\sqrt{x}+5)\frac{2}{3}\cdot\frac{3}{2}\frac{1}{\sqrt{x}}dx \\
&= 6\int \sec^2 u du.
\end{aligned}$$

We apply the appropriate formula from (20.1) getting

$$6\tan u + C = 6\tan\left(3\sqrt{x}+5\right) + C$$

This is the answer, which you should check by differentiating, but we shall not here. □

20.2.2 When substitution does not work

Sometimes the "obvious" choice of substitution does not work. This method of substitution allows you to see as soon as possible that it does not work, and you must look for an alternative method.

Example 108 *Find*

$$\int x(2x^3+5)^2dx.$$

Solution: We see we have a quantity to a power, so (as in Example 104) we try,

$$u = 2x^3+5, \qquad du = 6x^2dx.$$

But right here we see we cannot get x^2 since we only have an x. Thus we must try something else. The thing to do here is to square out the quantity and then multiply through with the x.

$$\begin{aligned}
\int x(2x^3+5)^2dx &= \int x(4x^6+20x^3+25)dx \\
&= \int \left(4x^7+20x^4+25x\right)dx \\
&= \frac{1}{2}x^8+4x^5+\frac{25}{2}x^2+C.
\end{aligned}$$

□

Example 109 *Find*

a) $\int \frac{x^3}{1+x^4}dx$,

b) $\int \frac{x}{1+x^4}dx$.

Solution:

a) We have a quantity in the denominator, so we try

$$u = 1 + x^4, \qquad du = 4x^3 dx$$

We see with the x^3 in the numerator we can get du, and we proceed to do it.

$$\begin{aligned} \int \frac{x^3}{1+x^4}dx &= \frac{1}{4}\int \frac{1}{1+x^4}\frac{d}{dx}\left(x^4\right)dx = \frac{1}{4}\int \frac{1}{u}du \\ &= \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln\left(1+x^4\right) + C \end{aligned}$$

b) If we were to try $u = 1+x^4$ again, we see with only an x in the numerator, we cannot get $du = 4x^3dx$. In searching for an alternate approach, we might ask, how do we make use of the single x in the numerator? If we observe x is the crucial part of the derivative of x^2, and that in the denominator $x^4 = (x^2)^2$, then we solve the problem with the surprising substitution

$$u = x^2, \qquad du = 2xdx.$$

Then

$$\begin{aligned} \int \frac{x}{1+x^4}dx &= \frac{1}{2}\int \frac{1}{1+(x^2)^2}\frac{d}{dx}\left(x^2\right)dx \\ &= \frac{1}{2}\int \frac{du}{1+u^2} + C = \frac{1}{2}\arctan(x^2) + C. \end{aligned}$$

□

20.3 Very Famous Example

Example 110 (Area of a circle) *Verify that the area of a circle of radius a is πa^2.*

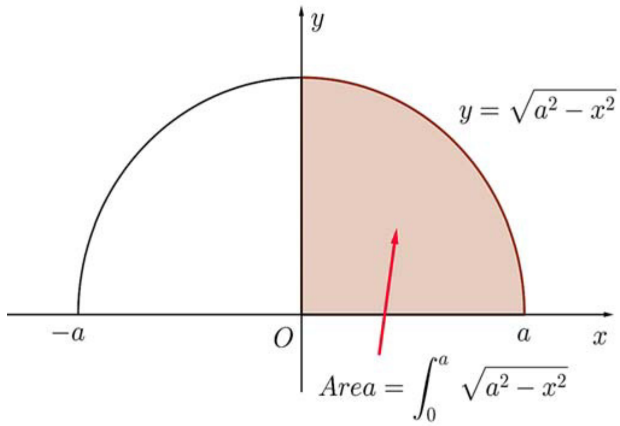


Fig. 20.1. A quarter-circle.

Solution: The function $f(x) = \sqrt{a^2 - x^2}$ describes the upper half of a circle centered at the origin with radius a (Figure 20.1). The region under this curve on the interval $[0, a]$ is a quarter-circle. Therefore, the area of the full circle is

$$4 \int_0^a \sqrt{a^2 - x^2} dx$$

Because the integrand contains the expression $a^2 - x^2$, we use the trigonometric substitution $x = a \sin\theta$. As with all substitutions, the differential associated with the substitution must be computed

$$x = a\sin\theta \quad \text{implies that} \quad dx = a\cos\theta.$$

Notice that the new variable θ plays the role of an angle.

The substitution works nicely, because when x is replaced by $a\sin\theta$ in the integrand, we have

$$\begin{aligned} \sqrt{a^2 - x^2} &= \sqrt{a^2 - (a\sin\theta)^2} \\ &= \sqrt{a^2\left(1 - \sin^2\theta\right)} \\ &= \sqrt{a^2\cos^2\theta} \\ &= |a\cos\theta| \\ &= a\cos\theta. \end{aligned}$$

We also change the limits of integration: When $x = 0$, $\theta = \arcsin 0 = 0$; when $x = a$, $\theta = \arcsin(a/a) = \arcsin(1) = \pi/2$. Making these substitutions, the

integral is evaluated as follows:

$$\begin{aligned}
4\int_0^a \sqrt{a^2-x^2}dx &= 4\int_0^{\pi/2} a\cos\theta\cdot a\cos\theta d\theta \\
&= 4a^2\int_0^{\pi/2}\cos^2\theta d\theta \\
&= 4a^2\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)|_0^{\pi/2} \\
&= \pi a^2.
\end{aligned}$$

20.4 Exercises

Exercise 20.1 *Evaluate the following indefinite integrals:*

1. $\int\left(3x^8+2x+x^{-3}\right)dx$
2. $\int\left(8x^2+3x^{-4}+x^{-8}\right)dx$
3. $\int\left(e^x+2x\right)dx$
4. $\int\left(e^{-2x}+8x^2\right)dx$
5. $\int\left(\sin 2x+3x\right)dx$
6. $\int\left(\cos 3x-2x+1\right)dx$
7. $\int\left(e^{-x}+2\cos x+5x^2\right)dx$
8. $\int\left(e^{3x}-8\sin 2x+x^{-4}\right)dx$

Exercise 20.2 *Evaluate each of the integrals in by making the indicated substitution, and check your answers by differentiating*

1. $\int 2x(x^2+4)^{3/2}dx; \qquad u=x^2+4$
2. $\int(x+1)(x^2+2x-4)^{-4}dx; \qquad u=x^2+2x-4$
3. $\int\frac{2y^7+1}{(y^8+4y-1)^2}dy; \qquad u=y^8+4y-1$
4. $\int\frac{x}{1+x^4}dx; \qquad u=x^2$
5. $\int\frac{\sec^3\theta}{\tan^3\theta}d\theta; \qquad u=\tan\theta$

6. $\int \tan x dx$; $u = \cos x$

Exercise 20.3 *Evaluate the following definite integrals:*

1. $\int_{-1}^{1} \sqrt{x+2}dx$
2. $\int_{2}^{3} \frac{dt}{t-1}$
3. $\int_{0}^{2} x\sqrt{x^2+1}dx$
4. $\int_{2}^{4} t\sqrt{t^2+1}dt$
5. $\int_{0}^{1} xe^{x^2}dx$
6. $\int_{0}^{1} \dfrac{e^x}{1+e^{2x}}dx$
7. $\int_{0}^{\pi/6} (\sin 3\theta + \pi)\, d\theta$
8. $\int_{0}^{\pi} \sin(\theta/2 + \pi/4)d\theta$
9. $\int_{0}^{\pi/2} \sin x \cos x dx$
10. $\int_{\pi/4}^{\pi/2} \cot \theta d\theta$

Exercise 20.4 *Evaluate the following indefinite integrals:*

1. $\int (x+1)\cos x dx$
2. $\int x \cos(5x)\, dx$
3. $\int x^2 \cos x dx$
4. $\int (x+2)\, e^x dx$
5. $\int \ln(10x)\, dx$
6. $\int x^2 \ln x dx$
7. $\int x^2 e^{3x} dx$
8. $\int \dfrac{1}{x^3} \cos \dfrac{1}{x} dx$
9. $\int x \ln x dx$

10. $\int \ln(9+x^2)dx$

Exercise 20.5 *Evaluate the definite integrals:*

1. $\int_1^2 x \ln x dx$

2. $\int_1^3 \ln x^3 dx$

3. $\int_0^1 xe^x dx$

4. $\int_1^e (\ln x)^2 \, dx$

5. $\int_0^{\pi/2} \sin(2x) \cos x dx$

6. $\int_{-\pi}^{\pi} e^{2x} \sin(2x) \, dx$

7. $\int_0^1 x \arctan x dx$

8. $\int_0^1 \frac{x^3}{\sqrt{x^2+1}} dx$

9. $\int_0^{\pi/2} (8+5x)\sin(5x) dx$

10. $\int_0^1 \arccos\left(\sqrt{y}\right) dy$

Exercise 20.6 (Integrating by parts) *Match the integral on the left, the correct substitution in the center and the evaluated integral on the right:*

$\int x \cos x dx$	$u = e^x, \quad v = -\frac{1}{4}\cos 4x$	$-\frac{1}{27}e^{-3x}\left(9x^2+6x+2\right)+C$
$\int x \sin 4x dx$	$u = x^2, \quad v = -\frac{1}{3}e^{-3x}$	$\frac{1}{2}\ln^2 x + C$
$\int x e^{2x} dx$	$u = x, \quad v = -\frac{1}{4}\cos 4x$	$\frac{1}{16}\sin 4x - \frac{1}{4}x\cos 4x + C$
$\int x \ln x dx$	$u = \ln x, \quad v = \ln x$	$\frac{1}{3}e^{x^3} + C$
$\int x^2 \ln x dx$	$u = x, \quad v = \sin x$	$\cos x + x\sin x + C$
$\int \frac{\ln x}{x} dx$	$u = \ln x, \quad v = \frac{1}{2}x^2$	$\frac{1}{17}e^x \sin 4x - \frac{4}{17}e^x\cos 4x + C$
$\int x^2 e^{-3x} dx$	$u = x, \quad v = \frac{1}{2}e^{2x}$	$\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$
$\int x^2 e^{x^3} dx$	$u = \ln x, \quad v = \frac{1}{3}x^3$	$\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$
$\int e^x \sin 4x dx$	$u = x^3, \quad v = 3x^2$	$\frac{1}{4}e^{2x}\left(2x-1\right)+C$

Exercise 20.7 (Integrating by parts) *Match the integral on the left, the correct substitution in the center and the evaluated integral on the right:*

$\int e^{2x}\cos x dx$	$u = \sin x, \quad v = -\frac{1}{2}\cos 2x$	$x\left(\ln^2 x - 2\ln x + 2\right) + C$
$\int \cos x \cos 2x dx$	$u = (\ln x)^2, \quad v = x$	$\frac{1}{5}e^{2x}\left(2\cos x + \sin x\right) + C$
$\int \sin x \sin 2x dx =$	$u = x^2, \quad v =$	$\frac{1}{2}\sin x - \frac{1}{6}\sin 3x + C$
$\int 4\arccos x dx =$	$u = e^{2x}, \quad v = \sin x$	$\frac{1}{27}e^{3x}\left(9x^2 - 6x + 2\right) + C$
$\int x^3 e^{x^2} dx$	$u = \ln(\sin x), \quad v = \sin x$	$(\sin x)\left(\ln(\sin x) - 1\right) + C$
$\int (\ln x)^2 dx$	$u = arc\cos x, \quad v = x$	$\frac{1}{2}e^{x^2}\left(x^2 - 1\right) + C$
$\int x^2 e^{3x} dx$	$u = x^2, \quad v = \frac{1}{2}e^{x^2}$	$\frac{1}{2}\sin x + \frac{1}{6}\sin 3x + C$
$\int \cos x \ln(\sin x) dx =$	$u = \cos x, \quad v = \frac{1}{2}\sin 2x$	$4x\arccos x - 4\sqrt{1 - x^2} + C$
$\int x \sin x^2 dx$	$u = x^2, \quad v = \frac{1}{3}e^{3x}$	$-\frac{1}{2}\cos x^2 + C$

Exercise 20.8 (Integrating by parts) *Match the integral on the left, the correct substitution in the center and the evaluated integral on the right:*

$\int_0^1 x\sin 2x dx$	$u = 2x, \quad v = \sin x$	$10\ln 10 - 10$
$\int_0^\pi 2x\cos x dx$	$u = x, \quad v = -\frac{1}{x}\cos \pi x$	$2\ln 2 - \frac{3}{4}$
$\int_0^1 x\cos \pi x dx$	$u = x, \quad v = \frac{1}{2\pi}\sin 2\pi x$	$\frac{1}{4}\sin 2 - \frac{1}{2}\cos 2$
$\int_0^1 xe^{3x} dx$	$u = \ln x, \quad v = \frac{1}{2}x^2$	0
$\int_0^1 x\cos 2\pi x dx$	$u = x, \quad v = \frac{1}{\pi}\sin \pi x$	$-\frac{2}{\pi^2}$
$\int_0^{10} \ln x dx$	$u = x, \quad v = \frac{1}{3}e^{3x}$	$\frac{2}{9}e^3 + \frac{1}{9}$
$\int_1^2 x\ln x dx$	$u = \ln x, \quad v = x$	-4
$\int_0^1 x\sin \pi x dx$	$u =, \quad v =$	$\frac{1}{\pi}$

21

Techniques of integration, part two (Exercises)

21.1 Trigonometric Substitutions

Trigonometric substitution is a technique for converting integrands to trigonometric integrals. For example, we can use the substitution $x = a\sin\theta$ for integrands containing radicals such as $a^2 - x^2$. These substitutions convert an integral in the original variable x to an integral in the new variable θ. This new integral will involve trigonometric functions. After evaluating the new integral, you convert your answer back to the original variable. You can conveniently represent these substitutions by right triangles, as shown in the examples. For definite integrals, you can also convert the limits of integration to the new variable θ , thus avoiding the need to return to the original variable.

21.1.1 The Sine Substitution

A sine substitution is motivated by the identity

$$1 - \sin^2\theta = \cos^2\theta.$$

If we define θ according to

$$x = \sin\theta,$$

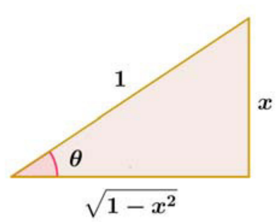
then notice, that

$$dx = \cos\theta d\theta$$

and

$$1 - x^2 = \cos^2\theta.$$

The relationship between the variables x and θ may be represented by a right triangle



From this right triangle we can easily read off all of the six trigonometric functions of θ in terms of x.

Example 111 *Find the integral*

$$\int \sqrt{1-x^2}dx.$$

Solution: When we replace x with $\sin\theta$, we have

$$\begin{aligned}\int \sqrt{1-x^2}dx &= \int \sqrt{\cos^2\theta}\cos\theta d\theta \\ &= \int \cos^2\theta d\theta = \frac{1}{2}\int (1+\cos 2\theta)\, d\theta \\ &= \frac{1}{2}(\theta+\sin\theta\cos\theta)+C\end{aligned}$$

Now we are ready to express the the result in terms of the original variable x :

$$\int \sqrt{1-x^2}dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right)+C.$$

□

Example 112 *Find the integral*

$$\int \frac{\sqrt{4-x^2}}{x}dx.$$

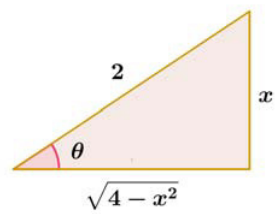
Solution: When we replace x with $2\sin\theta$, we have

$$\begin{aligned}x &= 2\sin\theta \\ dx &= 2\cos\theta d\theta\end{aligned}$$

and

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x}dx &= \int \frac{\sqrt{4\cos^2\theta}}{2\sin\theta}2\cos\theta d\theta \\ &= 2\int \frac{\cos^2\theta}{\sin\theta}d\theta = 2\int \frac{1-\sin^2\theta}{\sin\theta}d\theta \\ &= 2\int (\csc\theta - \sin\theta)\, d\theta \\ &= 2\ln|\csc\theta-\cot\theta| + 2\cos\theta + C\end{aligned}$$

In terms of the original variable x



$$\int \frac{\sqrt{4-x^2}}{x}dx = 2\ln\left|2-\sqrt{4-x^2}\right| - 2\ln|x| + \sqrt{4-x^2} + C.$$

□

21.1.2 The Tangent Substitution

A tangent substitution is motivated by the identity

$$1+\tan^2\theta = \sec^2\theta.$$

If we define θ according to

$$x = \tan\theta$$

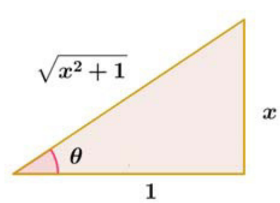
then

$$dx = \sec^2\theta d\theta$$

and

$$1+x^2 = \sec^2\theta.$$

The relationship between the variables x and θ may be represented by a right triangle



From this right triangle we can easily read off all of the six trigonometric functions of θ in terms of x.

Example 113 *Evaluate*

$$\int \frac{1}{(1+x^2)^2}dx$$

Solution: When we replace x with $\tan\theta$, we have

$$\begin{aligned} dx &= \sec^2\theta \\ 1+x^2 &= \sec^2\theta \end{aligned}$$

and

$$\begin{aligned} \int \frac{1}{(1+x^2)^2}dx &= \int \frac{1}{\sec^4\theta}\sec^2\theta d\theta \\ &= \int \cos^2\theta d\theta \\ &= \frac{1}{2}(\theta+\sin\theta\cos\theta)+C. \end{aligned}$$

Now we are ready to express the result in terms of the original variable x

$$\begin{aligned} \int \frac{1}{(1+x^2)^2}dx &= \frac{1}{2}\left(\arctan x+\frac{x}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+x^2}}\right)+C \\ &= \frac{1}{2}\left(\arctan x+\frac{x}{1+x^2}\right)+C \end{aligned}$$

□

Now we can consider a slightly more complicated example.

Example 114 *Find*

$$\int \sqrt{9+4x^2}dx$$

Solution:

$$\begin{aligned} 2x &= 3\tan\theta \\ dx &= \frac{3}{2}\sec^2\theta d\theta \\ 9+4x^2 &= 9\sec^2\theta \end{aligned}$$

So

$$\begin{aligned} \int \sqrt{9+4x^2}dx &= \int \sqrt{9\sec^2\theta}\frac{3}{2}\sec^2\theta d\theta = \frac{9}{2}\int \sec^3\theta d\theta \\ &= \frac{9}{2}\cdot\frac{1}{2}\left(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right)+C. \end{aligned}$$

Now, in terms of the original variable x, we have

$$\begin{aligned}
&\int \sqrt{9+4x^2}dx \\
=\ & \frac{9}{4}\left(\frac{1}{3}\sqrt{9+4x^2}\cdot\frac{2}{3}x+\ln\left|\frac{1}{3}\sqrt{9+4x^2}+\frac{2}{3}x\right|\right)+C \\
=\ & \frac{9}{4}\left(\frac{2}{9}x\sqrt{9+4x^2}+\ln\left|\sqrt{9+4x^2}+2x\right|-\ln 3\right)+C \\
=\ & \frac{1}{4}\left(2x\sqrt{9+4x^2}+9\ln\left|\sqrt{9+4x^2}+2x\right|\right)+C.
\end{aligned}$$

□

The Secant Substitution

A secant substitution is motivated by the identity

$$\sec^2\theta - 1 = \tan^2\theta.$$

If we define θ according to

$$x = \sec\theta,$$

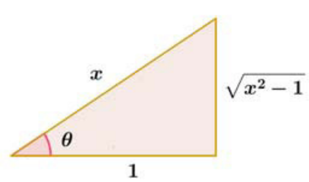
then notice that

$$dx = \sec\theta\tan\theta d\theta$$

and

$$x^2 - 1 = \tan^2\theta.$$

The relationship between the variables x and θ may be represented by an associated right triangle



From this right triangle we can easily read off all of the six trigonometric functions of θ in terms of x.

Example 115 *Evaluate*

$$\int \frac{1}{x^2\sqrt{x^2-1}}dx.$$

Solution: As

$$\begin{aligned} x &= \sec\theta \\ dx &= \sec\theta\tan\theta d\theta \\ x^2-1 &= \tan^2\theta, \end{aligned}$$

we have

$$\begin{aligned} \int \frac{1}{x^2\sqrt{x^2-1}}dx &= \int \frac{\sec\theta\tan\theta}{\sec^2\theta\tan\theta}d\theta \\ &= \int \frac{1}{\sec\theta}d\theta = \int \cos\theta d\theta \\ &= \sin\theta + C \\ &= \frac{\sqrt{x^2-1}}{x} + C. \end{aligned}$$

□

Let us consider slightly more complicated example:

Example 116 *Find*

$$\int \frac{x}{x\left(x^2-2\right)^{3/2}}dx.$$

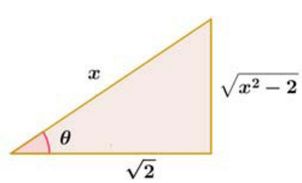
Solution: Here we want to substitute

$$x = \sqrt{2}\sec\theta.$$

That makes

$$\begin{aligned} dx &= \sqrt{2}\sec\theta\tan\theta d\theta, \\ x^2-2 &= 2\sec^2\theta - 2 = 2\tan^2\theta. \end{aligned}$$

The right triangle associated with that substitution is shown below:



Now let us tale the integral

$$\begin{aligned}
\int \frac{x}{x\left(x^2-2\right)^{3/2}} dx &= \int \frac{\sqrt{2}\sec\theta\tan\theta d\theta}{\sqrt{2}\sec\theta \cdot 2\sqrt{2}\tan^3\theta} \\
&= \frac{1}{2\sqrt{2}} \int \frac{1}{\tan^2\theta} d\theta \\
&= \frac{1}{2\sqrt{2}} \int \cot^2\theta d\theta \\
&= \frac{1}{2\sqrt{2}} \int \left(\csc^2\theta - 1\right) d\theta \\
&= \frac{1}{2\sqrt{2}} \left(-\cot\theta - \theta\right) + C,
\end{aligned}$$

and finally

$$\int \frac{x}{x\left(x^2-2\right)^{3/2}} dx = -\frac{1}{2\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{x^2-2}} + \operatorname{arcsec} \frac{x}{\sqrt{2}} \right) + C$$

□

21.1.3 Completing a square

Now let us consider an example which involves more general quadratic expression. Let us find the integral

$$\int \frac{x}{\sqrt{5-4x-x^2}}.$$

First let's complete a square

$$5-4x-x^2 = 5-\left(x^2+4x+4\right)+4 = 9-(x+2)^2$$

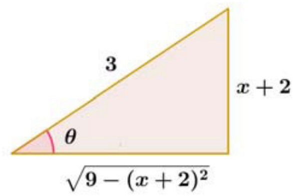
which gives

$$\int \frac{x}{\sqrt{5-4x-x^2}} = \int \frac{x}{\sqrt{9-(x+2)^2}}.$$

Now, we can use the substitution

$$\begin{aligned}
x+2 &= 3\sin\theta \\
dx &= 3\cos\theta d\theta \\
9-(x+2)^2 &= 9\cos^2\theta.
\end{aligned}$$

The relationship between the variables x and θ may be represented by an associated right triangle of the following form:



Now, we can find that

$$\begin{aligned}
\int \frac{x}{\sqrt{9-(x+2)^2}} &= \int \frac{3\sin\theta - 2}{3\cos\theta} 3\cos\theta d\theta \\
&= \int (3\sin\theta - 2)\, d\theta \\
&= -3\cos\theta - 2\theta + C \\
&= -\sqrt{9-(x+2)^2} - 2\arcsin\left(\frac{x+2}{3}\right) + C.
\end{aligned}$$

Example 117 (An Application of Arc Length) *A thin wire is in the shape of the parabola* $y = \frac{1}{2}x^2$, $0 \leq x \leq 1$. *What is the length of the wire?*

Solution: The derivative is $y' = x$, and hence, the formula for arc length gives

$$s = \int_0^1 \sqrt{1+(y')^2} dx = \int_0^1 \sqrt{1+x^2} dx.$$

We then use the trigonometric substitution

$$x = \tan\theta,\ dx = \sec^2\theta d\theta,\ \text{and hence } \sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sec\sqrt{1+x^2}.$$

The indefinite integral becomes

$$\int_0^1 \sqrt{1+x^2} dx = \int \sec\theta \left(\sec^2\theta\right) d\theta = \int \sec^3\theta d\theta.$$

Rather than return to the variable x , we can use the equation $x = \tan\theta$ to convert the limits of integration:

When $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \pi/4$. Hence, we have the interesting conclusion that

$$s = \int_0^1 \sqrt{1+x^2} dx = \int_0^{\pi/2} \sec^3\theta d\theta.$$

Using integration by parts twice, you obtain the formula for the right-hand integral, and we have the final answer:

$$\begin{aligned} s &= \frac{1}{2}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_0^{\pi/4} \\ &= \frac{1}{2}\ln\left(\sqrt{2}+1\right) + \frac{1}{2}\sqrt{2} \approx 1.147\,8 \end{aligned}$$

□

21.2 L'Hopital's method and partial fractions

We would love to be able to integrate any rational function

$$f(x) = \frac{p(x)}{q(x)}$$

where p, q are polynomials. This is where **partial fractions** come in. The partial fraction method writes $p(x)/q(x)$ as a sum of functions of the type which we can integrate. This is an algebra problem. Here is an important special case: In order to integrate

$$\int \frac{1}{(x-a)(x-b)}dx$$

write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

and solve for A, B.

In order to solve for A, B, write the right hand side as one fraction again

$$\frac{A}{(x-a)} + \frac{B}{(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$$

We only need to look at the nominator:

$$1 = Ax - Ab + Bx - Ba$$

In order that this is true we must have $A + B = 0$, $Ab - Ba = 1$. This allows us to solve for A, B.

Example 118 *To integrate*

$$\int \frac{2}{1-x^2}dx$$

we can write

$$\frac{2}{1-x^2} = \frac{1}{1+x} + \frac{1}{1-x}$$

and integrate each term

$$\int \frac{2}{1-x^2} dx = \ln(1+x) - \ln(1-x).$$

$\square$

Example 119 *Integrate*

$$\int \frac{5-2x}{x^2-5x+6} dx$$

Solution: The denominator is factored as $(x-2)(x-3)$. Write

$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}.$$

Now multiply out and solve for A, B:

$$A(x-2) + B(x-3) = 5 - 2x.$$

This gives the equations $A + B = -2$, $-2A - 3B = 5$. From the first equation we get $A = -B - 2$ and from the second equation we get $2B + 4 - 3B = 5$ so that $B = -1$ and so $A = -1$. So, we have obtained

$$\frac{5-2x}{x^2-5x+6} = -\frac{1}{x-3} - \frac{1}{x-2}$$

and can integrate:

$$\int \frac{5-2x}{x^2-5x+6} dx = -\ln|x-3| - \ln|x-2| + C.$$

Actually, we could have got this one also with substitution. How? $\square$

Example 120 *Integrate*

$$\int \frac{1}{1-4x^2} dx$$

Solution: The denominator is factored as $(1-2x)(1+2x)$. Write

$$\frac{1}{1-4x^2} = \frac{A}{1-2x} + \frac{B}{1+2x}.$$

We get $A = 1/4$ and $B = -1/4$ and get the integral

$$\int f(x) dx = \frac{1}{4}\ln|1-2x| - \frac{1}{4}\ln|1+2x| + C \quad \square$$

There is a fast method to get the coefficients based on L'Hospital's rule

> If a is different from b, then the coefficients A, B in
>
> $$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$
>
> are
>
> $$A = \lim_{x\to a}(x-a)f(x) = \frac{p(a)}{(a-b)}, \quad B = \lim_{x\to b}(x-b)f(x) = \frac{p(b)}{(b-a)}.$$

Proof. If we multiply the identity with $x-a$ we get

$$\frac{p(x)}{(x-b)} = A + \frac{B(x-a)}{x-b}$$

Now we can take the limit $x \to a$ without peril and end up with

$$A = p(a)/(a-b).$$

■

Cool, isn't it? This **L'Hopital's method can save you a lot of time!** Especially when you deal with more factors and where sometimes complicated systems of linear equations would have to be solved. Remember

> Math is all about elegance and does not use complicated methods if simple ones areavailable.

Example 121 *Find the anti-derivative of*

$$f(x) = \frac{2x+3}{(x-4)(x+8)}$$

Solution: We write

$$\frac{2x+3}{(x-4)(x+8)} = \frac{A}{x-4} + \frac{B}{x+8}$$

Now $A = \frac{2*4+3}{4+8} = \frac{11}{12}$, and $B = \frac{2*(-8)+3}{(-8-4)} = \frac{13}{12}$. We have

$$\frac{2x+3}{(x-4)(x+8)} = \frac{11/12}{x-4} + \frac{13/12}{x+8}$$

The integral is

$$\frac{11}{12}\ln|x-4| + \frac{13}{12}\ln|x+8| + C$$

□

Here is an example with three factors:

Example 122 *Find the anti-derivative of*

$$f(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}.$$

Solution: We write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Now

$$\begin{aligned} A &= \frac{1^2+1+1}{(1-2)(1-3)} = \frac{3}{2}, \quad B = \frac{2^2+2+1}{(2-1)(2-3)} = -7, \\ C &= \frac{3^2+3+1}{(3-1)(3-2)} = \frac{13}{2} \end{aligned}$$

The integral is

$$\frac{3}{2}\ln(x-1) - 7\ln(x-2) + \frac{13}{2}\ln(x-3) + C$$

□

21.3 Some famous examples

Example 123 (Circumference of a circle) *Confirm that the circumference of a circle of radius a is $2\pi a$.*

Solution: The upper half of a circle of radius a centered at $(0,0)$ is given by the function $f(x) = \sqrt{a^2 - x^2}$ for $|x| \le a$.

So we might consider using the arc length formula on the interval $[-a, a]$ to find the length of a semicircle. However, the circle has vertical tangent lines at $x = \pm a$ and $f'(\pm a)$ is undefined, which prevents us from using the arc length formula. An alternative approach is to use symmetry and avoid the points $x = \pm a$. For example, let's compute the length of one-eighth of the circle on the interval $[0, a/\sqrt{2}]$ (Figure 21.1).

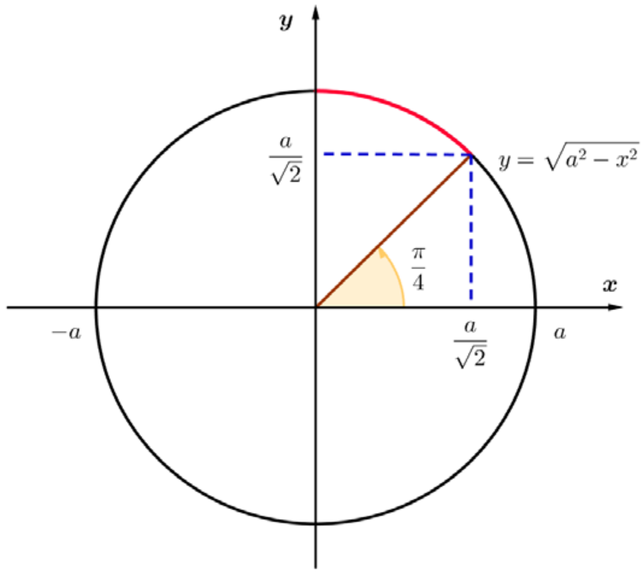


Fig. 21.1. One eight of the circle.

We first determine that

$$f'(x) = \frac{x}{\sqrt{a^2 - x^2}},$$

which is continuous on $[0, a/\sqrt{2}]$. The length of one-eighth of the circle is

$$\begin{aligned}
& \int_0^{a/\sqrt{2}} \sqrt{1 + (f'(x))^2}dx \\
= & \int_0^{a/\sqrt{2}} \sqrt{1 + \left(\frac{x}{\sqrt{a^2 - x^2}}\right)^2}dx \\
= & \; a\int_0^{a/\sqrt{2}} \frac{dx}{\sqrt{a^2 - x^2}} \\
= & \; a \arcsin \frac{x}{a}|_0^{a/\sqrt{2}} \\
= & \; a \arcsin \frac{1}{\sqrt{2}} = \frac{\pi a}{4}
\end{aligned}$$

It follows that the circumference of the full circle is $8\frac{\pi a}{4} = 2\pi a$ units.

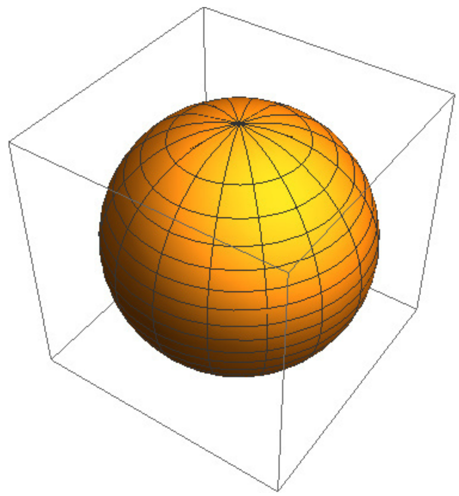


Fig. 21.2. A sphere

Example 124 *Find the area of the spherical surface of radius r obtained by revolving the graph of $y = \sqrt{r^2 - x^2}$ on $[-r, r]$ about the x axis (see Figure 21.2).*

Solution: We have

$$\sqrt{r^2 - [f'(x)]^2} = \frac{r}{\sqrt{r^2 - x^2}},$$

so the area is

$$\int_{-r}^{r} 2\pi\sqrt{r^2 - x^2}\frac{r}{\sqrt{r^2 - x^2}}dx = 2\pi r\int_{-r}^{r} dx = 4\pi r^2$$

which is the usual value for the area of a sphere.
□

Example 125 *The disk with radius* 1 *and center* $(4, 0)$ *is revolved around they* y *axis. Sketch the resulting solid and find its volume*[1]*.*

Solution: The doughnut-shaped solid is shown in Fig. 21.3. We observe that if the solid is sliced in half by a plane through the origin perpendicular to they y axis, the top half is the solid obtained by revolving about they y axis the region under the semicircle

$$y = \sqrt{1 - (x - 4)^2}$$

[1] Mathematicians call this a *solid torus*. The surface of this solid (an "inner tube") is a *torus*.

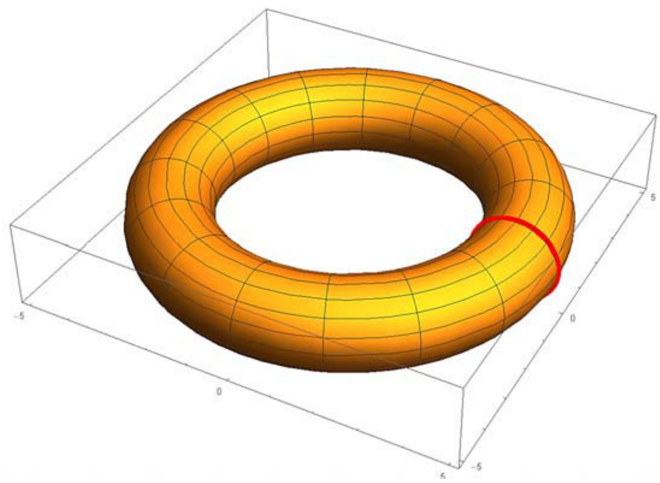


Fig. 21.3. The disk $(x-4)^2 + y^2 \leq 1$ is revolved about they axis.

on the interval $[3, 5]$. The volume of that solid is

$$\begin{aligned} V &= 2\pi \int_3^5 x\sqrt{1-(x-4)^2}dx \\ &= 2\pi \int_{-1}^1 (u+4)\sqrt{1-u^2}du \qquad (u = x-4) \\ &= 2\pi \int_{-1}^1 u\sqrt{1-u^2}du + 8\pi \int_{-1}^1 \sqrt{1-u^2}du \end{aligned}$$

Now $\int_{-1}^1 u\sqrt{1-u^2}du = 0$ because the function $f(u) = u\sqrt{1-u^2}$ is odd.

On the other hand, $\int_{-1}^1 \sqrt{1-u^2}du$ is just the area of a semicircular region of radius 1-that is, $\pi/2$-so the volume of the upper half of the doughnut is $8\pi \cdot (\pi/2) = 4\pi^2$, and the volume of the entire doughnut is twice that, or $8\pi^2$.

(Notice that this is equal to the area π of the rotated disk times the circumference 8π of the circle traced out by its center $(4, 0)$.) $\square$

21.4 Exercises

Exercise 21.1 *Evaluate the following integrals:*

1. $\int \sin^3 x \cos^3 x dx$,
2. $\int \sin^2 x \cos^5 x dx$,

3. $\int_0^{2\pi} \sin^4 x dx$,

4. $\int_0^{\pi/2} \sin^2 x \cos^4 x dx$,

5. $\int_0^{\pi/4} \sin^2 x \cos 2x dx$,

6. $\int \cos 2x \sin x dx$,

7. $\int \sin 4x \sin 2x dx$,

8. $\int_0^{2\pi} \sin 5x \sin 2x dx$.

Exercise 21.2 *Evaluate the following integrals:*

1. $\int \frac{\sqrt{x^2-4}}{x} dx$,

2. $\int \frac{\sqrt{x^2-9}}{x} dx$,

3. $\int \sqrt{1-u^2} du$,

4. $\int \sqrt{9-16t^2} dt$,

5. $\int \frac{s}{\sqrt{4+s^2}} ds$,

6. $\int \frac{x^3}{\sqrt{x^2-1}} dx$,

7. $\int \frac{x}{\sqrt{4x^2+x+1}} dx$,

8. $\int \frac{dx}{\sqrt{3+2x-x^2}}$.

Exercise 21.3 *Evaluate the following integrals:*

1. $\int \frac{1}{(x-2)(x^2+1)} dx$

2. $\int \frac{1}{(x-2)^2(x^2+1)} dx$

3. $\int \frac{1}{(x-2)^2(x^2+1)^2} dx$

4. $\int \frac{x^4+2x^3+3}{(x-4)^6}dx$

5. $\int \frac{x^2}{(x-2)}dx$

6. $\int \frac{1}{(x-2)(x^2+2x+2)}dx$

7. $\int \frac{1}{(x-2)(x^2+3x+2)}dx$

8. $\int \frac{x^3+1}{x^3-1}dx$

9. $\int \frac{1}{8x^3+1}dx$

10. $\int \frac{x}{x^4+2x^2+2}dx$

Exercise 21.4 *Use L'Hospital's method to find*

a) $\int \frac{23}{(x+2)(x-3)(x-2)(x+3)}dx$

b) $\int \frac{1}{(x+1)(x-1)(x+7)(x-3)}dx$

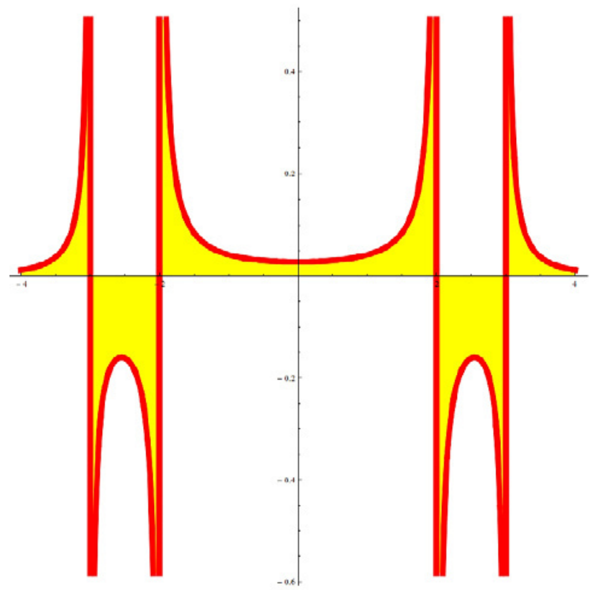
Exercise 21.5 *Match the integral on the left, and the evaluated integral on the right:*

$\int \cos x \sin^4 x dx$	$\frac{1}{48}$
$\int \cos^2 x \sin^4 x dx$	$\frac{1}{16}x - \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x + \frac{1}{192}\sin 6x$
$\int_0^{\pi/4} \cos x \sin^5 x dx$	$\frac{1}{3}$
$\int_{\pi/4}^{\pi/3} \cos^3 x \sin^3 x dx$	$-\frac{1}{4}$
$\int_0^{\pi/2} \cos^2 x \sin x dx$	$\frac{1}{2}x + \frac{1}{4}\sin 2x$
$\int_{-\pi/2}^{0} \cos^3 x \sin x dx$	$\frac{3}{8}x - \frac{3}{16}\pi - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$
$\int \cos^2 x dx$	$\frac{1}{8}\sin x - \frac{1}{16}\sin 3x + \frac{1}{80}\sin 5x$
$\int \sin^4 x dx$	$\frac{11}{384}$

Exercise 21.6 *Integrate*

$$\int_{-1}^{1} \frac{1}{(x+3)(x+2)(x-2)(x-3)} dx.$$

The graph of the function is shown to the below. Lets call it the friendship graph.



Exercise 21.7 *"One,Two,Three,Four Five, once I caught a fish alive!"*

$$\int \frac{(1+2x+3x^2+4x^3+5x^4)}{(1+x+x^2+x^3+x^4+x^5)} dx = ?$$

Exercise 21.8 *Find*

$$\int (1+x+x^2+x^3+x^4)(\sin(x)+e^x) dx$$

Exercise 21.9 *Find*

$$\int \log(x) \frac{1}{x^2} dx.$$

Exercise 21.10 *Find the following definite integral*

$$\int_1^2 x + \tan(x) + \sin(x) + \cos(x) + \log(x) dx.$$

Exercise 21.11 *Evaluate the following integral:*

$$\int_0^{2\pi} \sin(\sin(\sin(\sin(\sin(x)))))dx$$

Explain the answer you get.

Exercise 21.12 An evil integral*: Evaluate*

$$\int \frac{1}{x\log(x)}dx$$

`HINT`*: Can you figure out a function $f(x)$ which has $\dfrac{1}{x\log(x)}$ as the derivative?*

22

Taylor polynomials (Exercises)

22.1 Practice Problems

Example 126 *Find Taylor polynomial* $P_4(x)$ *for* $\ln(1+x)$.

Solution: Let $f(x) = \ln(1+x)$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$\begin{array}{ll} & f(0) = 0 \\ f'(x) = \dfrac{1}{x+1} & f'(0) = 1 \\ f''(x) = -\frac{1}{(x+1)^2} & f''(0) = -1 \\ f'''(x) = \frac{2}{(x+1)^3} & f'''(0) = 2 \\ f^{(4)}(x) = -\frac{6}{(x+1)^4} & f^{(4)}(0) = -6 \end{array}$$

Therefore

$$\begin{aligned} P_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 0 + x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \end{aligned}$$

(see Figure 22.1).

Example 127 *Find Taylor polynomial of the function* $g(x) = \arctan(x)$ *for* $a = 1$ *and* $n = 3$ *and give the Lagrange form of the remainder.*

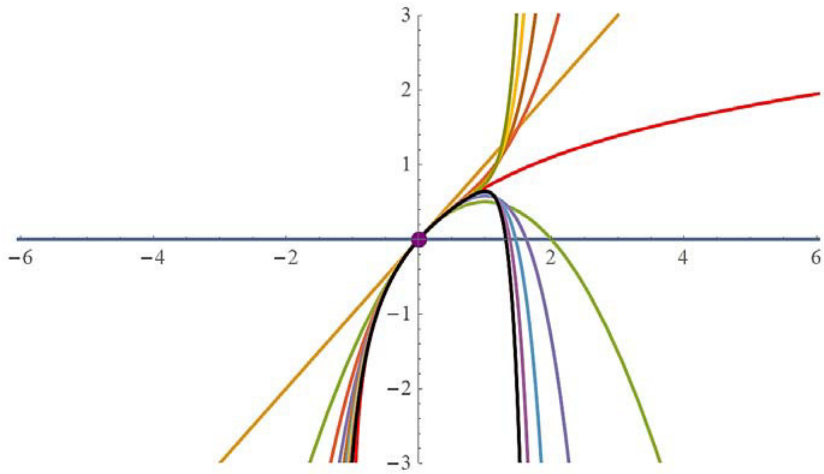


Fig. 22.1. Taylor polynomials for $\ln(1+x)$

$$g(1) = \tfrac{1}{4}\pi$$

$$g'(x) = \frac{1}{x^2+1} \qquad g'(1) = \frac{1}{2}$$

$$g''(x) = -\frac{2x}{\left(x^2+1\right)^2} \qquad g''(1) = -\frac{1}{2}$$

$$\begin{aligned} g'''(x) &= 8\tfrac{x^2}{(x^2+1)^3} - \frac{2}{\left(x^2+1\right)^2} \qquad g'''(1) = \frac{1}{2} \\ &= \frac{\left(6x^2-2\right)}{\left(x^2+1\right)^3} \end{aligned}$$

Therefore,

$$\begin{aligned} P_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= \frac{1}{4}\pi + \frac{1}{2}(x-1) - \frac{\frac{1}{2}}{2!}(x-1)^2 + \frac{\frac{1}{2}}{3!}(x-1)^3 \\ &= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 . \end{aligned}$$

The Lagrange form of the remainder is

$$R_4(x) = \frac{g^{(4)}(c)}{4!}(x-1)^4\,,$$

where c is between 1 and π. We need to find the fourth derivative of the given function. From the work above, we have

$$g'''(x) = \frac{\left(6x^2-2\right)}{\left(x^2+1\right)^3},$$

so,

$$\begin{aligned} g^{(4)}(x) &= 24\frac{x}{\left(x^2+1\right)^3} - 48\frac{x^3}{\left(x^2+1\right)^4} \\ &= -24x\frac{x^2-1}{\left(x^2+1\right)^4}. \end{aligned}$$

Therefore,

$$\begin{aligned} R_4(x) &= \frac{g^{(4)}(c)}{4!}(x-1)^4 \\ &= \frac{-24c\frac{1-c^2}{\left(c^2+1\right)^4}}{4!}(x-1)^4 \\ &= c\frac{1-c^2}{\left(c^2+1\right)^4}(x-1)^4 \end{aligned}$$

where c is between 1 and π. $\square$

Appendixes

A1. Greek letters used in mathematics, science, and engineering

The Greek letter forms used in mathematics are often different from those used in Greek-language text: they are designed to be used in isolation, not connected to other letters, and some use variant forms which are not normally used in current Greek typography. The table below shows Greek letters rendered in TEX

Table 22.1. Greek letters used in mathematics

α		alpha	ν		nu
β		beta	ξ	Ξ	xi
γ	Γ	gamma	π	Π	pi
δ	Δ	delta	ρ		rho
ϵ		epsilon	σ	Σ	sigma
ζ		zeta	τ		tau
η		eta	υ		upsilon
θ	Θ	theta	ϕ	Φ	phi
ι		iota	χ		chi
κ		kappa	ψ	Ψ	psi
λ	Λ	lambda	ω	Ω	omega
μ		mu	$\dagger$		dagger

TEXis a typesetting system designed and mostly written by Donald Knuth at Stanford and released in 1978.

Together with the Metafont language for font description and the Computer Modern family of typefaces, TeX was designed with two main goals in mind: to allow anybody to produce high-quality books using a reasonably minimal amount of effort, and to provide a system that would give exactly the same results on all computers, now and in the future.

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